The Bell System Technical Journal

October, 1928

The Practical Application of the Fourier Integral [1]

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ABSTRACT: The growing practical importance of transients and other non-periodic phenomena makes it desirable to simplify the application of the Fourier integral in particular problems of this kind and to extend the range of problems which can be solved in closed form by this method. Unless the physicist or technician is in a position to evaluate the definite integrals which occur, by mechanical means, he is usually entirely dependent upon the results obtained by the professional mathematician. To facilitate the use of the known closed form evaluations of Fourier integrals many of them have been compiled and tabulated in Table I. They are presented, however, not as definite integrals but as paired functions, one function being the coefficient for the cisoidal oscillation (or complex exponential) and the other function the reciprocally related coefficient for the unit impulse. This arrangement simplifies the table and promises to be most convenient in practical applications, since it is the coefficients of which immediate use is made, just as in the case of the Fourier series. Applications of the tabulated coefficient pairs to 85 transient problems are given, together with all necessary details, in Table II.

INTRODUCTION

THE Fourier integral and the Fourier series are alternative expressions of the Fourier theorem, the series being a limiting case of the integral and vice versa. Usually the theorem is approached from the side of the series, but there are also advantages in the approach from the integral side, which is the method followed in this paper. The generality and importance of the theorem is well expressed by Kelvin and Tait who said: ". . . Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance." For any real understanding of the theorem it is necessary to appreciate why it is one of the most beautiful mathematical results and why it furnishes an indispensable instrument in physics.

The Fourier integral is a most beautiful mathematical result because of the economy of means employed in obtaining a most general result.

[1] Presented September 13, 1927, in preliminary form at the International Congress of Telegraphy and Telephony in Commemoration of Volta.

One form of integral is used both to analyze and to synthesize. In both cases it is the product of the arbitrary function and the elementary sinusoidal oscillation which is integrated. This achieves the mathematical counterpart of spectrum analysis and spectrum synthesis. The functions resulting from analysis and synthesis stand in a mutually reciprocal relation.[2] They are paired with each other. Either of these functions may be assigned with an astonishing degree of arbitrariness. Singular cases being excepted, the mate function is then determined uniquely and definitely by the integral. While the sine, cosine and complex exponential are most commonly used as the elementary expansion functions, an entire class of functions present the same fundamental relations and find applications in the more recondite problems.

The Fourier integral is an indispensable instrument in connection with physical systems in which cause and effect are linearly related (so that the principle of superposition holds) because it gives at once an explicit formal solution of general problems in terms of the solution for the sinusoidal case which is often readily found. This explicit general solution makes use of two Fourier integrals, one for the spectrum analysis of the arbitrary cause and the other for the spectrum synthesis of the component sinusoidal solutions. No further consideration of the actual physical system is necessary after the elementary sinusoidal solution has been obtained. This point of view has become a part of our general background of thought.

Unfortunately the actual evaluation of specific Fourier integrals in closed form presents formidable if not insuperable difficulties. Only a small number of distinct general integrals have been evaluated in closed form in the century and more which has elapsed since the Fourier integral discovery was announced. Additions to the list of evaluated Fourier integrals can ordinarily be made only by the professional mathematician. Unless the physicist or technician is in a position to evaluate Fourier integrals by mechanical means, or is satisfied to employ infinite series or other infinite processes in place of the definite integrals, he is usually entirely dependent upon the evaluations which the professional mathematician has made in the past or is able to make for his special use. On this account, it is often desirable to so formulate practical problems that only evaluated Fourier integrals will occur. It would be well for the physicist and technician to become familiar with the Fourier integral evaluations which the professional mathematician has achieved.

[2] The fundamental importance of the Fourier integral may be associated with an analogy which exists between the integral and the imaginary unit, both considered as operators. In both cases two iterations of the operation merely change a sign and four iterations completely restore the original function.

It is the purpose of this paper to take the first steps towards the preparation of two tables, one giving the evaluations of Fourier integrals and the other giving the sinusoidal solutions for physical systems. Together they would reduce the practical application of the Fourier integral to the selection of three results from these two tables. Thus by means of the first table the arbitrary cause could be resolved into a sum of sinusoidal causes; by means of the second table the solutions for these sinusoidal causes could be supplied; and, finally, by means of the first table again, the effect of superposing these sinusoidal solutions could be shown, and thus the answer to the original problem would be given.

The preparation of the tables calls primarily for a compilation of the results already obtained by pure analysis, after which new evaluations and new solutions should be added, in so far as is possible. No attempt has yet been made to completely cover the existing literature on the subject, which extends back over one hundred years and is extensive and widely scattered. But sufficient has been done to show that the forms of the tables which are proposed are most convenient for practical application.

Paired Coefficients—Terminology

The Fourier integral theorem has been expressed in several slightly different forms to better adapt it for particular applications. It has been recognized, almost from the start, however, that the form which best combines mathematical simplicity and complete generality makes use of the exponential oscillating function $e^{i2\pi ft}$. More recently the overwhelming advantage of using this oscillating function in the discussion of sinusoidal oscillatory systems has been generally recognized. It is, therefore, plain that this oscillating function should be adopted as the basic oscillation for both of the proposed tables. A name for this oscillation, associating it with sines and cosines, rather than with the real exponential function, seems desirable. The abbreviation cis x for $(\cos x + i \sin x)$ suggests that we name this function a cis or a cisoidal oscillation. This term is tentatively employed in this paper. The notation $\mathrm{cis}(2\pi ft)$ is also employed where it is desired to use an expression which is essentially one-valued, which avoids the use of exponentials, or which suggests periodic oscillations by its connection with cosine and sine.[3]

[3] Since the cisoidal oscillation is simply a uniform rotation at unit distance about the origin in the complex plane, it may be desirable to try some compact notation which directly suggests this rotation; for example, $\mathrm{ru}(ft)$, 1^{ft}, i^{4ft} might be defined as the complex quantity obtained by rotating unity through ft complete turns or $4ft$ quadrants.

In a table of Fourier integrals, every integral expression would then contain, in addition to the arbitrary function $F(f)$, the same oscillating function $\text{cis}(2\pi ft)$, the same integral sign with limits $-\infty$, $+\infty$ and the same differential df. To repeat any such group of a dozen characters in each of several hundred entries seems quite unnecessary. It is, therefore, proposed merely to tabulate the arbitrary function $F(f)$ and the value $G(t)$ for the evaluated integral expressed as a function of the time. The table is thereby reduced to two parallel columns of associated functions, one of which is employed as the coefficient of the elementary cisoidal function while the other is a function of the independent time variable. The table would, however, be more symmetrical if both of the associated functions could be regarded as coefficients of an elementary function. This may be done by introducing the unit impulse as an elementary function, the impulse occurring at the epoch g at which instant it presents a unit area whereas its value is zero for all time before and all time after the epoch g. This is an essentially singular function and to recognize this fact it will be designated by $\mathfrak{S}_0(t-g)$ which is intended to emphasize the singularity. The time function may now be replaced in the table by the same function of the parameter g, since the time function $G(t)$ is equal to the integral with respect to g between infinite limits of the product $G(g)\mathfrak{S}_0(t-g)$.

The table of Fourier integrals has now become also a table of paired coefficient functions. This means that if the coefficient $F(f)$ is employed with the cisoid, and the coefficient $G(g)$ is employed with the unit impulse, and both products are summed for the entire infinite range of their parameters f and g, the same identical resulting time function is obtained.[4] Taken in connection with their respective elementary functions, the two associated coefficient functions are, therefore, equivalent, alternative ways of representing a particular time function. This is the fundamental geometrical or physical point of view which is needed in connection with the practical application of the Fourier integral theorem. For this reason the table has been headed a table of Paired Coefficients; as explained above, however, it may equally well be considered to be a table of Fourier Integrals.

There is another fundamental reason for placing both of the functions $F(f)$ and $G(g)$ on the same footing as coefficients. It is this:

[4] The use of frequency and epoch as the two parametric variables gives us many symmetrical formulas where, if the radian frequency were employed, an unsymmetrical 2π would occur. In practical applications the frequency of the coefficient pairs becomes the frequency which is ordinarily employed in acoustics, in music and in commercial alternating currents. The basic unit for frequency is the reciprocal second; the unit for epoch is the second.

Fourier's fundamental discovery was that the two functions may be transposed in the Fourier integral if the sign of one of the parameters is reversed. Thus, either one of the two functions constituting any coefficient pair may be taken as the coefficient of the cisoidal oscillation, provided only that the proper sign is given the epoch parameter occurring in the other function. For this reason also both functions are thus quite properly regarded as coefficients.

It is found convenient to call each coefficient of a coefficient pair the mate of the other coefficient, pair and mate being employed just as in the case of gloves. To find the mate of a glove, it is necessary to know all about the given glove including the fact as to whether it is the right or the left one of the pair. In the same way, to find the mate of a coefficient function, it is necessary to know not only the form of the function, but, in addition, whether its variable is the frequency or the epoch. The notation $\mathfrak{M}G(g)$, $\mathfrak{M}F(f)$ will be employed to indicate the mate of the particular coefficient $G(g)$, $F(f)$.

We have now defined and explained the proposed terminology for use in the practical application of the Fourier integral theorem. Before proceeding to practical applications, it is desirable to become familiar with these coefficient pairs considered in their own right. We may well begin by reminding the reader of the dissimilarity between the elementary oscillations.

The Two Elementary Functions Contrasted

The dissimilarity between the two elementary functions of the time, the cisoidal oscillation $\mathrm{cis}(2\pi ft)$ and the unit impulse $\mathfrak{S}_0(t-g)$ is most striking. This is clearly shown by the wire models of Fig. 1 where each function is depicted for five values of its parameter. For the value zero the cisoidal oscillation degenerates into an infinite straight line parallel to the time axis and cutting the real axis at $x = 1$. For the same value zero of its parameter g, the unit impulse is zero everywhere except at the origin where it has a vanishingly narrow loop extending to $x = +\infty$.

For other values of the parameter, the cisoidal oscillation is always an infinite cylindrical helix, centered on the time axis, and passing through the point $x = 1$, while the infinite loop of the impulse function is displaced unchanged along the time axis to $t = g$. For positive values of the parameter f, the cisoidal oscillation is a right-handed helix, with pitch equal to f^{-1}, and thus decreasing as f increases. For negative values of f, the pitch is the same but the helix is left-handed.

Both functions have essential singularities, which are quite different both in character and in location. For the cisoidal oscillation the singularity is always located at $t = \infty$; for the impulse the singularity is at $t = g$.

The fundamental differences between the two elementary time functions adapt them for different uses. It is desirable to be in a position to employ first one and then the other, shifting from one to the other without any trouble or delay, so that at each step of a problem the elementary function best suited for use may be employed.

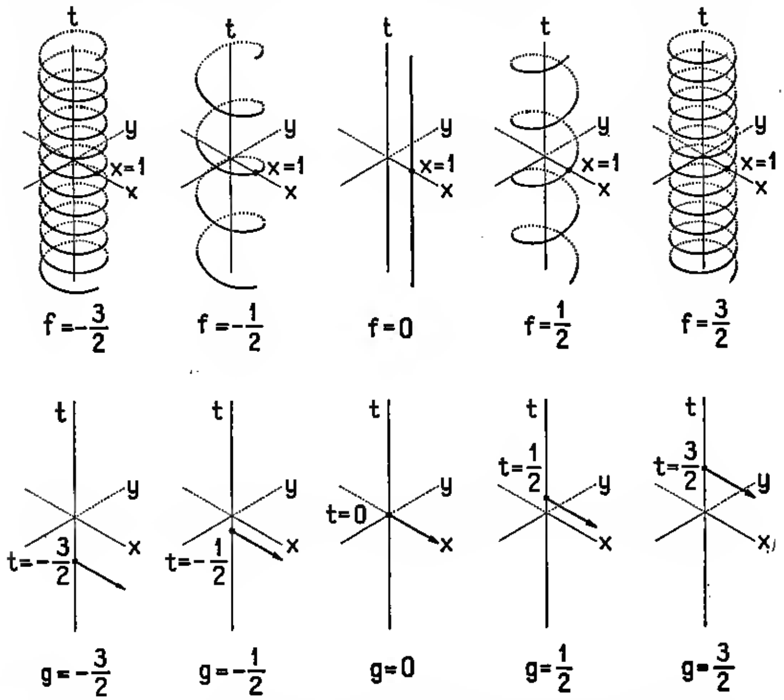

Fig. 1—Wire models of cisoidal oscillations cis $(2\pi ft)$ (above) and of unit impulses $\mathfrak{S}_0(t - g)$ (below) for the particular values 0, $\pm 1/2$, $\pm 3/2$, of the parameters f and g.

For this we require only an adequate table of pairs and a certain familiarity in the use of the pairs. It is desirable to acquire the habit of thinking of the coefficients of a pair as alternative representations of a curve.

The Use of Table I for Obtaining Coefficient Pairs[5]

The table is divided into nine parts. In Part 1 are given the general processes for deriving any coefficient mate; but such processes are to

[5] Five other closely related uses may be made of Table I as explained in the first footnote to that table. Operational expressions are brought within the scope of the table by substituting for the operator $p = d/dg$ the particular value $i2\pi f$, other possible interpretations of the operator, if any, being ignored.

be employed only when it is necessary to start from first principles. All mates which have once been determined may be taken from the latter sections of the table with a great saving of time and energy. Part 2 of the table shows the elementary transformations and combinations of pairs; these theorems may be employed either to extend a given table of coefficient pairs, or to cover a given group of coefficients with a shorter table of specific pairs. It is assumed that anyone desiring to make serious use of the table will first become familiar with these elementary combinations and transformations; even the simple addition, factor and transposition theorems (201), (205), (217) are most useful.

Part 3 of the table contains seven pairs, which are called key pairs because all specific pairs listed in the entire table may apparently be derived from them by specialization or by passing to a limit after any necessary use has been made of the elementary combinations and transformations of Part 2, amplified, as indicated, by the removal of certain unnecessary restrictions to real quantities. If an assigned coefficient is not included in Part 3 as thus generalized, then this coefficient cannot be found anywhere in Table I. Part 3, therefore, serves the useful purpose of giving a bird's-eye view of the entire table. The seven pairs are presumably redundant as they stand.

For applicational purposes it is most desirable to have a table which lists the precise pair required; many special cases which have been used in practical applications may be found in Parts 4–9 of Table I which constitute a short classified list of particular cases. It is important to remember that a given coefficient should be looked for on the other side of the table if it is not found on its own side since all pairs are transposable by (217) or (218). In the tables as they stand, some pairs have been transposed, but this is not true in the majority of cases.

Whenever an infinite process is to be employed, such as infinite series, integration or differentiation, the permissibility of the process is a question which must be answered for the particular case in hand; the formal result given in Table I may break down, for example, if either the original or the transformed pair is a singular pair. This general warning necessarily applies to every part of the subject of coefficient pairs just because it is a part of the general subject of mathematical analysis.

It is intended that the statement of each pair in the entire table shall eventually include every limitation and every warning which the mathematical sponsors for that pair would consider necessary to guarantee its safe use by anyone understanding the fundamental nature of coefficient pairs. A beginning has been made by specifying the branches of multiple-valued functions and the method of approaching limits. When this has been fully carried out, any pair may be taken from the table and used without the least concern as to the analytical methods by which the validity of the pairs has been established. Thus the finished table will make possible a complete separation of the analytical evaluation of all known Fourier integrals from their practical applications.

Having now explained, in a general way, the use of Table I, it will be useful to consider in detail a limited number of the pairs which are of special practical interest.

General Processes for Deriving the Mate

The table is naturally headed by the two fundamental Fourier integrals (101), (102) because of their intrinsic importance as explicit and implicit definitions of coefficient mates. The chief purpose of the table, however, is to make it possible for the technical man to make the fullest use of coefficient pairs without concerning himself at all as to the analytical work of evaluating either of these Fourier integrals. Pairs (101) and (102) are thus intended to serve mainly as definitions for the pairs which follow.

The statement has been made that essentially only one Fourier integral has been evaluated by determining the indefinite integral and substituting the integration limits. Whether or not this is precisely true, the statement does illustrate the fact that the formulation of the Fourier integral does not in itself suggest a practical finite analytical process for the actual evaluation of the definite integral. No such system of evaluating definite integrals is known. Writing down the Fourier integral amounts to little more than definitely formulating a question.

If the coefficient $F(f)$ is expanded as a finite or infinite series in powers of f (or p), the mate is given by pair (106*), and this involves a finite or infinite series of essentially singular functions which are further considered below in connection with Fig. 3. If a series expansion of $F(f)$ is made in terms of any functions of f for which the mates are known, there is a corresponding series for the mate. Some of these pairs are shown as (104*)–(112). The possibility of the formal infinite expansion does not necessarily imply the convergence of the series in the case of coefficient pairs any more than in other general developments.

The technical man is not ordinarily a master of infinite series, definite integrals or other infinite processes. It is, therefore, highly desirable to give him coefficient pairs which are in closed form, that is, involve only a finite number of operations with known functions. Accordingly, the portion of the table expressible in closed form has seemed to be the part which should be developed first. Specific pairs requiring infinite series for their expression have not been included in this preliminary draft of Table I. The omission of these series and of other infinite processes does not signify any failure to appreciate their importance. It is intended to include specific infinite series later.

The Elementary Transformations of Coefficient Pairs

The simple addition theorem (201) is of the greatest practical importance. The summation may include any number of pairs; they may be quite unrelated, or they may be the successive terms of power expansions as shown in (106*)–(111*). Next to the addition theorem we may place the multiplication theorem (202) or (203), special cases of which are of great practical importance. Among these special cases are (206)–(211) where any coefficient is multiplied or divided by its parameter or by a cisoidal oscillation of its parameter.

Any real linear substitution for the frequency and epoch parameters is made possible by the simple transformations (205)–(207), (214). The generalization of these transformations by the removal of the restriction to real numbers is allowable in important cases as is indicated by the parameters shown in square brackets with each pair of Part 3.

The differentiation and integration of coefficients with respect to the frequency, epoch or other parameter give the important transformations (208)–(213).

Some of the simple transformations continue to yield new results when they are repeated any number of times or when several transformations are combined in sequence. Pairs (216), (218)–(222) are examples of such combinations. All pairs in Parts 4–9 of this table may apparently be derived from the seven key pairs of Part 3 by means of these transformations employing complex parameters as indicated in Part 3, and passage to a limit in certain cases.

The resolution of pairs into the four types of i^n-multiple pairs, as shown by pairs (223)–(225), throws considerable light on the nature of coefficient pairs.

Some of the elementary properties of pairs are expressed in words as follows:

Elementary Properties of Pairs

(1) The sum or difference of pairs is a pair. Cf. pair (201).

(2) Any constant multiple of a pair is also a pair. Cf. pair (204).

(3) Any linear combination of pairs is also a pair. Cf. pairs (201), (204).

(4) The odd and even parts of every pair are also pairs.

(5) If both coefficients of a pair are real, both are even.

(6) If a pair has one real and one pure imaginary coefficient, both are odd.

(7) If a coefficient is even and real, its mate is also even and real.

(8) If a coefficient is odd and real, its mate is odd and pure imaginary, and vice versa.

(9) If a coefficient is real, its mate has conjugate values for opposite values of its parameter and conversely. Cf. pair (216).

(10) The conjugates of the coefficients of a pair are also a pair provided the sign of either frequency f or epoch g is reversed. Cf. pair (215).

(11) A pair with the signs of both frequency f and epoch g reversed is also a pair. Cf. pair (214).

(12) The parameter of either coefficient may be multiplied by a positive real constant provided the other parameter and coefficient are each divided by the same constant. Cf. pair (205).

(13) Coefficients of a pair may be interchanged if, when interchanging the parameters, the sign of one parameter, either f or g, is reversed. Cf. pair (217).

(14) Any pair may be resolved uniquely into the sum of four pairs by pairing together: the even, real parts; the even, imaginary parts; the odd, real part of each coefficient with the odd, imaginary part of the other coefficient.

(15) A pair may have the form $(F(f), \lambda F(g))$ where the multiplier λ is constant, if and only if λ has one of the four unit values $(1, i, -1, -i)$. Such a pair is called an i^n-multiple pair. Cf. pair (223).

(16) Any i^n-multiple pair has both coefficients odd or even according as n is odd or even.

(17) Any i^n-multiple pair with complex coefficients may be resolved into two i^n-multiple pairs with coefficients which are real or pure imaginary.

(18) The coefficients of any two i^n-multiple pairs are orthogonal if the i^n multipliers are different.

(19) The coefficients of any four i^n-multiple pairs with different i^n multipliers are linearly independent.

(20) Any pair may be resolved uniquely into the sum of four i^n-multiple pairs; i.e., pairs of the form $F_n(f)$, $i^n F_n(g)$. Cf. pair (224).

(21) Any pair may be resolved uniquely into the sum of eight i^n-multiple pairs where $F_n(f)$ is real or pure imaginary. Cf. pair (225)

Pairs Based on the Normal Error Law

The identical pair (703), $\exp(-\pi f^2)$, $\exp(-\pi g^2)$, is one of the simplest pairs and may well serve as the starting point in the consideration of specific coefficient pairs. Each coefficient is the broad impulse of the normal error law. It is remarkable that identical coefficients of this simple form should produce the same identical function when associated with either the cisoidal oscillation or the very different unit impulse.

If the differential transformation (222), taking the upper signs, is applied to the normal error law pair (703), the infinite series of ϕ_n pairs (702) is obtained. Of these derived pairs, the first eight are written out as pairs (704)–(711). The cisoidal coefficients are alternately even and odd functions which oscillate in the neighborhood of the origin, each successive coefficient having an added half oscillation. The ϕ_n pair has $(n + 1)$ half oscillations. Beyond these oscillations, every coefficient in the infinite sequence decreases rapidly and asymptotically to zero in both directions. The mates of these cisoidal coefficients are identically the same except for a constant coefficient which is i^n and thus goes cyclically through the four values, 1, i, -1, $-i$.

The $\phi_n(x)$ functions are shown by Fig. 2. They are essentially the parabolic cylinder functions of order n. These coefficients may be used for the expansion of every function which, with its first two derivatives, is continuous for all positive and negative values of the variable and for which a certain integral exists. This expansion is known as the Gram-Charlier series, which appears in pair (112).

Starting again with the normal law of error pair (703) in the form (701) and setting $\rho = \frac{1}{4}\beta/\pi$, and applying the differential transformation (208) repeatedly, we obtain the infinite sequence of pairs (713) of which the first five are listed as pairs (714)–(718). The cisoidal coefficients are the successive integral powers of p multiplied by the normal error exponential. The impulse coefficients are essentially the ϕ_n functions multiplied by the normal error exponentials. These pair, are plotted in Fig. 3 for the special case $\beta = a^2 = 1$.

Both of the infinite series of pairs derived from the error function and shown in Figs. 2 and 3 are regular throughout, are nowhere infinites and vanish at infinity.

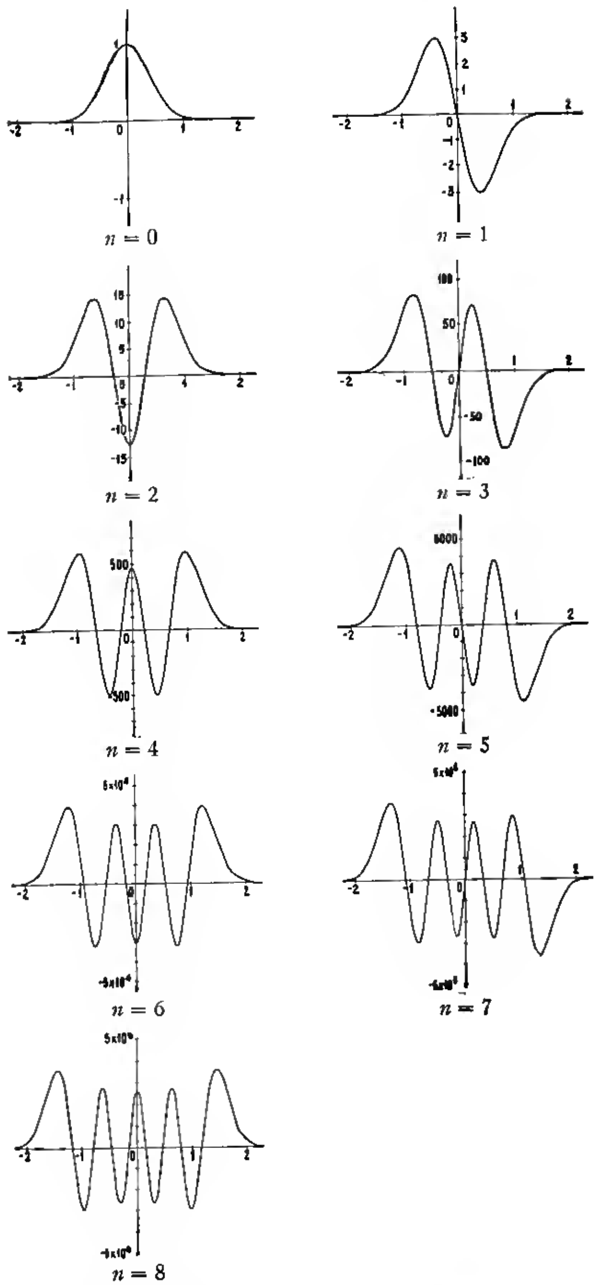

Fig. 2—Curves showing the $\phi_n(x)$ functions for $n = 0, 1, 2, \cdots, 8$
$\phi_n(x) = \exp(\pi x^2) D_x{}^n \exp(-2\pi x^2)$.

Essentially Singular Pairs for Integral Powers of the Parameter

If in Fig. 3, with the value of n held fixed, we allow a to approach the limit 0, the cisoidal coefficient becomes p^n and the impulse coefficient, which is compressed horizontally towards the origin and expanded vertically, with corresponding areas increasing as a^{-n}, ultimately vanishes everywhere except at the origin where it acquires an essential oscillating singular point. At the limit, then, a singular pair is obtained; it will be designated as p^n, $\mathfrak{S}_n(g)$. $\mathfrak{S}_n(g)$ is characterized by having all of its moments about the origin vanish except the nth moment, which is equal to $(-1)^n n!$ The dotted graphs on the left of Fig. 3 show p^n to the scales indicated. The curves on the right show $\mathfrak{S}_n(g)$ provided we assume that the horizontal scale is increased with a and the vertical scale increased inversely with a^{n+1} as a approaches the limit 0. Fig. 3 thus serves to picture the essentially singular function $\mathfrak{S}_n(g)$. That is, it is sufficient if the coefficient maintains this form while proceeding to the limit. This form is, however, not essential. It is necessary only that the method of approach to the limit give the same set of moments.

An alternative way of deriving the mate for the positive integral powers p^n is by means of a linear combination of $(n+1)$ pairs of the form of (603) with parameters equal to a, $2a$, $3a$, $\cdots$, $(n+1)a$, respectively, so that the first term in the power series expansion of the cisoidal coefficient is p^n. The corresponding impulse coefficient is a succession of $(n+1)$ bands, each of width a, the first band beginning at epoch zero, the heights of the successive bands being equal to the binomial coefficients for power n divided by a^{n+1} but alternately positive and negative. The mth moment of this impulse coefficient is 0 for $m < n$, equal to $(-1)^n n!$ for $m = n$, and proportional to a^{m-n} for $m > n$. Upon allowing a to approach zero, the cisoidal coefficient approaches p^n, and the impulse coefficient approaches $\mathfrak{S}_n(g)$, since in the limit the same set of moments is obtained as was found above to characterize the nth singularity function. This is pair (402*).

The special cases for $n = 0$, 1 are of most frequent occurrence. They are pairs (403*), (404*). $\mathfrak{S}_0$ is the unit impulse since its 0th moment equals unity; $\mathfrak{S}_1$ is the doublet with the moment -1 since its first moment is -1. $\mathfrak{S}_1$ and all higher order singular functions are included in the series coefficients of (104*), (106*).

Fig. 3 may be extended upward step by step from the normal error law pair by dividing by p on the left and integrating with respect to g on the right. At each step a constant of integration is introduced. The first two pairs thus obtained are pairs (725*) and (726*). Choos-

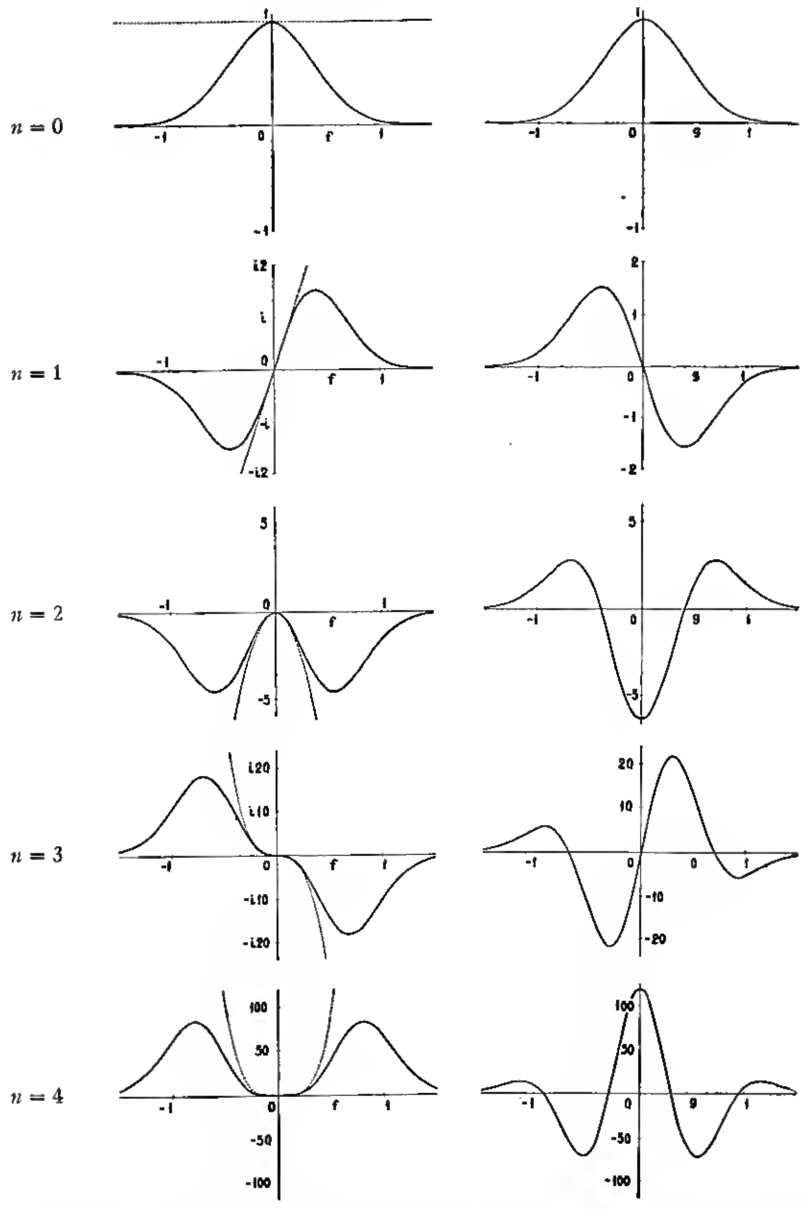

Fig. 3—Graphs for the family of pairs $p^n \exp(-\pi a^2 f^2)$ $a^{-1} D_g^n \exp(-\pi g^2/a^2)$. The heavy curves show the cases $a = 1$, $n = 0, 1, 2, 3, 4$; the dotted curves on the left are for the same values of n but for the limit $a \to 0$. On the right the curves apply for any value of a if the horizontal and vertical scales are multiplied by a and a^{-n-1} respectively.

ing the integration constants so as to make the impulse coefficients alternately odd and even, these two pairs are as shown in Fig. 4. If we now allow a to approach the limit zero, a new series of pairs is obtained of which the first two pairs are shown dotted in Fig. 4 for the particular choice of integration constants there made. The general limiting pair is designated as p^{-n}, $\mathfrak{S}_{-n}(g)$ and it is shown with its n arbitrary parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$ as pair (410*). In some ways it is simpler to derive the limiting pair for negative integral powers of p from rational functions of p, which may be accomplished as shown by pair (411*). Special cases are shown by pairs (408*), (409*), (415*), (416*).

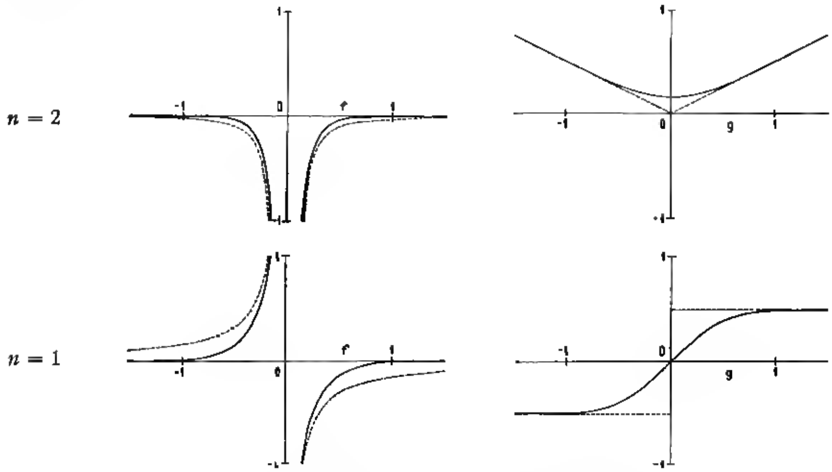

Fig. 4—Graphs for the family of pairs $p^{-n}\exp(-\pi a^2f^2)$, $a^{-1}D_g^{-n}\exp(-\pi g^2/a^2)$, with the integration constants chosen so as to make the impulse coefficients alternately odd and even. The heavy curves show the cases $a = 1$, $n = 1, 2$; the dotted curves show the limit $a \to 0$, $n = 1, 2$.

The first of the series $\mathfrak{S}_{-1}(g)$ is a unit step at epoch 0 from a constant value $\lambda - \frac{1}{2}$ for all negative epochs to the constant value $\lambda + \frac{1}{2}$ for all positive epochs. The constant λ may have any value; this is a singular case marked by the failure of the general rule that the choice of the cisoidal coefficient uniquely determines the impulse coefficient. This means that in any well set problem some other condition determines the value of the constant λ. In some problems, for example, it is necessary that the epoch coefficient be an odd function, and then λ vanishes. In other problems where either the epoch function must be zero for all negative epochs or on the other hand the p occurring in the cisoidal coefficient is actually the limit of $p + a$ as a approaches zero through positive values, the constant λ equals $\frac{1}{2}$. This limiting condition may arise if we assume that resistance may be ignored, as a first approximation, in studying actual systems which necessarily involve at least a small amount of dissipation.

The mates of positive and negative integral powers of p, including the zero power, cannot be derived directly and definitely from the Fourier integral (101) without the specification of an additional passage to a limit. Such pairs therefore differ essentially from the great body of regular pairs where the choice of one coefficient completely determines the mate. In order to permanently ear-mark these limiting pairs, their serial numbers in Table I bear a star. These pairs may be thought of as lying on the periphery of the great domain which includes the totality of regular pairs.

Identical Mates and Other Simply Related Mates

Since one of the coefficients of a pair may be assigned quite arbitrarily, this choice allows us, if we so elect, to specify some relation between the two coefficients of a pair. We might specify that a linear combination $\lambda F_j(x) + \mu G_j(x)$ of the two coefficients of a pair both taken with the parameter x is to equal an arbitrary function $F(x)$. The pair (F_j, G_j) is then uniquely determined, unless $\lambda + i^n\mu = 0$, being equal to pair (224) after each F_n has been divided by $\lambda + i^n\mu$. Again if it is specified that one coefficient is to be the reciprocal of the other, a possible solution is pair (760).

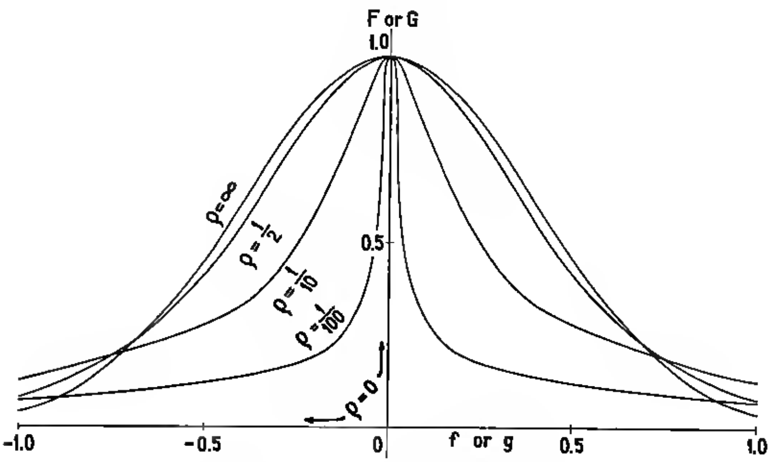

Fig. 5—Identical coefficient pairs of the form
$(1 + x^2/\rho^2)^{-\frac{1}{8}} K_{\frac{1}{4}}(2\pi\rho^2\sqrt{1 + x^2/\rho^2})/K_{\frac{1}{4}}(2\pi\rho^2), \qquad x = f \text{ or } g.$

The condition that the mates shall be identically the same function of their parametric variables f and g is of special interest. In addition to the identical pairs shown on Fig. 2, $n = 0, 4, 8$, the table contains a number of identical pairs including (523), (625), (712), (761), (916).

The identical pair (916) divided by its value at the origin is shown in Fig. 5 for different real values of its parameter ρ. For $\rho = +\infty$, the curve is of the $\exp(-\pi x^2)$ or normal law of error form, and is identical pair (703). For $\rho = \frac{1}{2}$, the reciprocal hyperbolic cosine identical pair (625) is shown correctly within the width of the line, this being apparently a mere coincidence since pair (916) does not include it as a special case. Finally, for $\rho = 0$, the limiting curve coincides with the horizontal axis taken together with unit length of the positive vertical axis. This represents pair (523) divided by its value at the origin, which is infinite. The point to be especially noted is that the area under every curve of the family illustrated by Fig. 5 is the same and equal to unity. This must hold for the limit $\rho = 0$, when the curve encloses no area within a finite distance of the origin.

The identical pair $|f^{-\frac{1}{2}}|$, $|g^{-\frac{1}{2}}|$ is of great simplicity and it occupies a central position among algebraic pairs. Starting with the minus one-half power of the parameters in both coefficients, any increase in the power of one parameter requires an equal decrease in the power of the other parameter as is illustrated, for example, by pairs (502*), (516*), (524).

It is not permissible to specify any relation whatsoever between the two coefficients of a pair; for example, no pair exists for which one coefficient is twice the other. As stated above, the only multiples permissible are the four units 1, i, -1, $-i$. For each of these four cases there are an infinite number of solutions. These solutions satisfy the integral equations given in the foot-note to pair (223).

Practical Applications of Coefficient Pairs

Fourier gave the first comprehensive method of finding the solution for transients. His method involves three steps: viz.,

I. Spectrum analysis of the cause among all frequencies.
II. Solution for all frequencies.
III. Spectrum synthesis of the effects for all frequencies.

Fourier thus substituted three problems for one. With a table of Fouricr coefficient pairs, these three steps may be made as follows:

I. Find the mate of the cause considered as an impulse coefficient.
II. Multiply this mate by the admittance for the system.
III. Find the matc of this product considered as a cisoidal coefficient.

These three steps define a perfectly definite result, since every arbitrarily chosen coefficient has a mate which is unique and determinate, or may be made so by the specification of some suitable passage to a limit.

The use of a table of pairs may also be stated in another and somewhat more general way as follows:

For any system where the principle of superposition holds, any cause $C(t)$, its effect $E(t)$ and the corresponding admittance $Y(f)$ are connected by a relation which may be written in any one of three ways which explicitly express each of the three quantities in terms of the remaining two, as follows:

$$E(g) = \mathfrak{M}[Y(f)\mathfrak{M}C(g)],$$

$$C(g) = \mathfrak{M}\left[\frac{\mathfrak{M}E(g)}{Y(f)}\right],$$

$$Y(f) = \frac{\mathfrak{M}E(g)}{\mathfrak{M}C(g)},$$

where $\mathfrak{M}$ is read "mate of."

The use of coefficient pairs may be most simply illustrated by reference to Figs. 3 and 4, in connection with the problem of finding transient currents through a perfect condenser of unit capacity due to impressed electromotive forces shown by each of the seven curves on the right considered as functions of the time. Any curve on the right being the cause, the next curve below it is the effect, considering Fig. 4 to be placed above Fig. 3. In the solution the first step is to find the mate of the curve on the right. This is the curve on the left. This mate is then to be multiplied by the admittance of the system which is p for a unit condenser. Reference to the titles of the figures shows that this product is given by the next lower curve on the left. To find the mate of this last curve is the third step in the solution and for this it is merely necessary to go to the curve on the right. The three steps then take us from any curve on the right to the next curve below it. Figs. 3 and 4, taken together, are a section of an infinite sequence of pairs which illustrate an infinite number of possible transients in a perfect condenser of unit capacity.

If, on the other hand, the system consisted of a perfect reactance coil of unit inductance and the impressed cause was again shown by any curve on the right, the effect would be shown by the next higher curve, assuming that the initial current at the beginning of time was that shown by the extreme left of the upper curve. Thus, when the cause is oscillating, there is one less half oscillation in the effect than in the cause. This is for an inductance. For a condenser, conditions are reversed; the effect has one more half oscillation than the cause.

The scales of Figs. 3 and 4 may be changed to correspond to any value of a, the parameter which appears in the coefficients of the pairs.

At the limit $a = 0$, the cause and effect would be the singular $\mathfrak{S}_n$ or $\mathfrak{S}_{-n}$ functions.

The curves on the right for $n = 0$ of Fig. 3 and $n = 1$ of Fig. 4 show that at the limit $a=0$ a unit step in the voltage produces a unit impulse in the current through a unit condenser; on the other hand, a unit impulse applied to a unit inductance gives a current which is a unit step.

The curves of Fig. 2 may be used to furnish another illustration of the use of coefficient pairs, in connection with the problem of finding networks in which assigned transient currents will be produced by assigned impressed electromotive forces. Any curve n being the assumed cause and the next curve $(n + 1)$ the assumed effect, the required admittance is $\phi_{n+1}(f)/[i\phi_n(f)]$. This admittance is presented by a ladder network of $(n + 1)$ elements: perfect inductance coils in the series arms, perfect condensers in the shunt arms, the ladder starting with a shunt condenser, the values of the shunt capacities being equal to 2, $2n(n - 1)^{-1}$, $2n(n - 1)^{-1}(n - 2)(n - 3)^{-1}$, etc., and the values of the series inductances being equal to $(2\pi n)^{-1}$, $(2\pi n)^{-1}(n - 1)(n - 2)^{-1}$, etc. In verifying the solution of this problem, it is to be noticed that the mates of the curves n and $(n + 1)$, regarded as impulse coefficients, are the same curves multiplied by i^{-n} and $i^{-(n+1)}$; the quotient of the latter mate divided by the former mate is the admittance of the network as given above.

On the other hand, any curve $(n + 1)$ being the cause, the curve n is the effect in the reciprocally related ladder network of $(n + 1)$ elements, starting with a series reactance coil, the values of the series inductances being equal to 2, $2n(n - 1)^{-1}$, $2n(n - 1)^{-1}(n - 2)(n - 3)^{-1}$, etc., and the values of the shunt capacities being equal to $(2\pi n)^{-1}$, $(2\pi n)^{-1}(n - 1)(n - 2)^{-1}$, etc.

Practical Applications of Coefficient Pairs in Table II.

In general, each of the three subsidiary problems employed by Fourier is unsolvable in closed form. In a strictly limited number of cases, however, all three problems have been solved and the final transient solution obtained. These solutions should be cherished and collected for ready reference. It is a needless waste of time to repeat the analytical work each time a solution is required. Except for a few special cases lying outside of the scope of the table, all practical applications of closed form coefficient pairs which were found in a preliminary search are included in the transient solutions of Table II. As it stands, the table is far from a complete list of closed form solutions, but it contains many important solutions and serves to illustrate the use of Table I. Table II contains 39 admittances, with references to 39 systems which serve to illustrate the occurrence of these admittances. In the third, fourth and fifth columns, 85 transient solutions are given of which 39 are for the unit impulse, 30 for the unit step, and 16 for the suddenly applied cisoid.

The causes producing the transients in Table II are but three in number: the unit impulse, the unit step, and the suddenly applied cisoid; and the mates for these causes are unity, p^{-1} and $(p - p_0)^{-1}$ as is shown by pairs (403*), (415*) and (440*). Multiplying these three mates by the admittances and taking the mates of the products, we have the effects, as is stated in the headings of the last three columns of the table.

To illustrate in detail the steps involved in finding a transient effect with the aid of Table I, consider system No. 14 of Table II with the cause equal to the unit step $\mathbf{S}_{-1}(t)$, $\lambda = \frac{1}{2}$. The mate of the unit step is p^{-1} by pair (415*). Multiplying this by $Y(f)$ as given in the second column of Table II, we have $up^{-\frac{1}{2}}(1 + \sqrt{p/\lambda})^{-1}$ for the cisoidal coefficient. By pair (551) the mate of this is $u\sqrt{\lambda}\exp(\lambda g)\operatorname{erfc}\sqrt{\lambda g}$, $0 < g$. Substituting for g the actual variable t, we have the transient solution as given in the fourth column and fourteenth row of Table II.

This simple example fully illustrates the three essential steps in finding any transient effect when the admittance and pairs are known. In this example the effect was considered to be the unknown. If either the cause or the admittance were the unknown, the same pairs would be involved but the two coefficients in a pair would be used in the reversed sequence in all but one instance.

There are still 32 squares of Table II left blank. It would be a simple matter to place series solutions or integral solutions in each of these squares. Thus if the impulse transient of column 3 is known, the other two transients are given at once in integral form by pairs (210) and (219); if the unit step transient of column 4 is known, the suddenly applied cisoidal transient is written immediately in integral form by the use of pair (220). The real problem is, however, either to find closed form solutions in terms of known functions or to show that this is impossible. When the failure of known functions has been established, we should next consider the choice of new functions so defined as to throw as much light as possible on the new solutions.

Table II may be regarded as another table of coefficient pairs. Column 2 contains cisoidal coefficients; column 3, the mates of these coefficients; column 4, the mates of these coefficients when multiplied by p^{-1}; and column 5, the mates of these coefficients when multiplied by $(p - p_0)^{-1}$. The corresponding pair in Table I is referred to in the lower left-hand corner of each square by its serial number. In a few cases, two or three pairs are referred to and there it is necessary to add the Table I pairs together or, in the case of systems 37–39, to apply the two pairs in sequence. In Table II, the customary physical notation is adhered to because it is often of long standing and this necessitates some change in notation when comparing pairs in the two tables.

Summary and Conclusions

Many practical applications of the Fourier integral have been simplified by the compilation of Tables I and II, which give coefficient pairs, admittances and transient solutions.

Minor changes in nomenclature and point of view have been introduced, all with the idea of simplifying the practical application of the Fourier integral, in the following ways:

(1) Using the cisoidal oscillation and the unit impulse side by side as alternative elementary expansion functions.

(2) Focusing attention upon coefficient pairs for these two elementary functions, both coefficients of a pair representing the resolution of the same arbitrary function.

(3) Using the frequency and epoch as the parametric variables, in place of the customary radian frequency and independent time variable.

(4) Employing as a coefficient any real or complex arbitrary function which may be practically useful by regarding it, where necessary, as a limit approached through coefficients which form regular pairs.

(5) Introducing the $\mathfrak{S}_n(g)$ functions having an essential oscillating singularity at the origin which mate with p^n, the positive integral powers of p.

(6) Using a notation which greatly reduces the number of occasions for employing the integral symbol in applications of the Fourier theorem.

Having established the inclusiveness and practical utility of the proposed coefficient pair method of applying the Fourier integral, we are now planning to critically verify the tables and make them as complete as is feasible. It is proposed to include eventually such references to the literature as may add to the interest of the tables. The contributions of integral equations and of the operational method to the present subject will also be incorporated in the tables. The preparation of similar tables for other elementary expansion functions, such as Bessel functions, is also a possibility. A comprehensive table might be made which would include in parallel columns the coefficient functions for a large number of elementary expansion functions, thus giving at once many alternative ways of representing particular time functions. This would make it possible to shift without trouble from any one expansion to any other expansion of the tabulation.

I am under great obligations to my colleagues for their contributions towards the preparation of this paper. I shall be grateful to any person who will call my attention to errors or omissions in any part of this paper.[6]

NOTATION

The following notation is employed in Table I; also in Table II, except as specifically restricted.

a, b, c	= positive reals.
br x	= branch x. For each multiple-valued function, branches are designated in one or more different ways. When no branch designation is given, branch zero is to be understood.
$C(z)$	$= \int_0^z \cos(\frac{1}{2}\pi z^2)dz = -C(-z). \quad C(\pm\infty) = \pm\frac{1}{2}.$
$\text{cis}(z)$	$= \cos z + i \sin z = \exp(iz) = e^{iz}$ = cisoidal oscillation if $z = 2\pi ft$.
$D_\nu(z)$	= parabolic cylinder function of order ν. $D_n(z) = \exp(-\frac{1}{4}z^2)H_n(z). \quad D_{-\frac{1}{2}}(z) = (2\pi)^{-\frac{1}{2}}z^{\frac{1}{2}}K_{\frac{1}{4}}(\frac{1}{4}z^2).$ $D_{-1}(z) = (\frac{1}{2}\pi)^{\frac{1}{2}}\exp(\frac{1}{4}z^2)\,\text{erfc}(2^{-\frac{1}{2}}z).$
$\text{erf}(z)$	$= \frac{2}{\sqrt{\pi}}\int_0^z \exp(-z^2)dz = -\text{erf}(-z). \quad \text{erf}(\pm\infty) = \pm 1.$
$\text{erfc}(z)$	$= \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-z^2)dz = 1 - \text{erf}(z).$
f	= frequency; parameter for the cisoidal oscillation. $-\infty < f < \infty$.
$F(f)$	= coefficient for cisoidal oscillation, parameter f.
$F_n(f)$	= coefficient of an i^n-multiple pair $(F_n(f), i^nF_n(g))$ in pairs (223)–(225).

[6] I am already much indebted to M. Paul Lévy for a number of suggestions including the expression of the general identical pair as the sum of any pair having even coefficients and its transposed pair.

g	= epoch; parameter for the unit impulse. $-\infty < g < \infty$.
$g_0 < g < g_1$	restricts the given coefficient to the indicated limits; outside these limits the coefficient is zero.
$G(g)$	= coefficient for unit impulse, parameter g.
$H_n(z)$	$= z^n - \frac{n(n-1)}{2} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} z^{n-4} - \cdots$
	= Hermite polynomial of order n.
$H_\nu^{(1)}(z)$	$= \frac{2}{\pi} i^{-\nu-1} K_\nu(-iz)$ = Bessel function of the third kind.
$H_\nu^{(2)}(z)$	$= \frac{2}{\pi} i^{\nu+1} K_\nu(iz)$ = Bessel function of the third kind.
$I(z)$	= imaginary part of z. $z = R(z) + iI(z)$.
$I_\nu(z)$	$= i^{-\nu} J_\nu(iz)$.
j, k, l	= integers greater than zero.
$J_\nu(z)$	$= i^\nu I_\nu(-iz)$ = Bessel function of the first kind.
$K_\nu(z)$	$= \frac{1}{2}\pi i^{\nu+1} H_\nu^{(1)}(iz)$.
m, n	= positive integers, including zero.
$\mathfrak{M}(\ )$	= mate of ( ).
p	$= i2\pi f$, the imaginary radian frequency.
r, s	= reals, positive or zero.
$R(z)$	= real part of z. $z = R(z) + iI(z)$.
$S(z)$	$= \int_0^z \sin(\frac{1}{2}\pi z^2)dz = -S(-z)$. $S(\pm\infty) = \pm\frac{1}{2}$.
$\mathfrak{S}_v(x)$	$= \lim_{a \to \infty} aD_x^v \exp(-\pi a^2 x^2)$ = vth singularity function.
$\mathfrak{S}_{-n}(x)$	$= \left(\lambda_1 \pm \frac{1}{2(n-1)!}\right) x^{n-1} + \lambda_2 x^{n-2} + \cdots + \lambda_n$, $0 < \pm x$, $0 < n$.
t	= time. $-\infty < t < \infty$.
v, w	= integers, positive, negative or zero.
x, y	= reals, unrestricted.
Y	= admittance of system for cisoidal oscillation.
$Y_\nu(z)$	$= \frac{1}{2}i[H_\nu^{(2)}(z) - H_\nu^{(1)}(z)]$ = Bessel function of the second kind.

z = complex quantity, unrestricted.

$\bar{z}$ = conjugate of z.

z^μ, br x = $\exp[\mu R(\log z) + i\mu \arg z]$, where $(2x - 1)\pi < \arg z \leqq (2x + 1)\pi$.

= $e^{i2\pi\mu v} z^\mu$, br$(x - v)$. Branches $(x + v)$, $v = 0, \pm 1, \pm 2, \cdots$ form a complete set and without repetition unless μ is a rational real.

$\alpha, \beta, \gamma, \delta$ = complex quantities, real parts greater than zero.

θ = principal argument. $-\pi < \theta \leqq \pi$.

λ, μ, ν = complex quantities, unrestricted.

ρ, σ, τ = complex quantities, real parts not less than zero.

$\phi_n(x)$ = $\exp(\pi x^2) D_x{}^n \exp(-2\pi x^2)$

= $(-2\pi^{\frac{1}{2}})^n D_n(2\pi^{\frac{1}{2}} x)$ where D_n is the parabolic cylinder function of order n

= $(-2\pi^{\frac{1}{2}})^n \exp(-\pi x^2) H_n(2\pi^{\frac{1}{2}} x)$ where H_n is the Hermite polynomial of order n.

$\psi(z)$ = $\Gamma'(z)/\Gamma(z)$ = logarithmic derivate of the gamma function. $-\psi(1)$ = Euler's constant = $0.5772\cdots$.

* marks a pair as being the limit approached by regular pairs.

	Not Restricted	Real Part $\geqq 0$	Real Part > 0
Integers	v, w	m, n	j, k, l
Reals	f, g, t, x, y	r, s	a, b, c
Complex	z, λ, μ, ν	ρ, σ, τ	$\alpha, \beta, \gamma, \delta$

TABLE I

Paired Coefficients for the Cisoidal Oscillation and the Unit Impulse[1]

Part 1. General Processes for Deriving the Mate

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation $\text{cis}(2\pi ft) = \exp(pt)$	Coefficient $G(g)$ for the Unit Impulse $\mathfrak{S}_0(t-g) = \lim_{a\to 0} \frac{1}{a} e^{-\pi(t-g)^2/a^2}$
101.	F	$\int_{-\infty}^{\infty} F(f)\text{cis}(2\pi fg)df$
102.	$\int_{-\infty}^{\infty} G(g)\text{cis}(-2\pi fg)dg$	G
103.	F	$D_g \int_{-\infty}^{\infty} F(f)\text{cis}(2\pi fg)p^{-1}df$
104.*	$\lambda_1(p-p_0) + \lambda_2(p-p_0)^2 + \cdots$, lim by 401*	$\text{cis}\,(2\pi f_0 g)[\lambda_1\mathfrak{S}_1(g) + \lambda_2\mathfrak{S}_2(g) + \cdots]$
105.*	$\lambda_1 \frac{1}{(p-p_0)} + \lambda_2 \frac{1}{(p-p_0)^2} + \cdots$, lim by 408*	$\text{cis}\,(2\pi f_0 g)\left(\lambda_1 + \lambda_2 \frac{g}{1!} + \lambda_3 \frac{g^2}{2!} + \lambda_4 \frac{g^3}{3!} + \cdots\right)$, $0 < g$
106.*	$\lambda_1 p + \lambda_2 p^2 + \lambda_3 p^3 + \cdots$, lim by 401*	$\lambda_1\mathfrak{S}_1(g) + \lambda_2\mathfrak{S}_2(g) + \lambda_3\mathfrak{S}_3(g) + \cdots$
107.*	$\lambda_1 \frac{1}{p} + \lambda_2 \frac{1}{p^2} + \lambda_3 \frac{1}{p^3} + \cdots$, lim by 408*	$\lambda_1 + \lambda_2 \frac{g}{1!} + \lambda_3 \frac{g^2}{2!} + \lambda_4 \frac{g^3}{3!} + \cdots$, $0<g$
108.*	$\frac{1}{p^a}\left(\lambda_0 + \lambda_1 \frac{1}{p} + \lambda_2 \frac{1}{p^2} + \cdots\right)$, $0 < a < 1$, lim by 516*	$\frac{1}{\Gamma(a)g^{1-a}}\left[\lambda_0 + \lambda_1 \frac{1}{a} g + \lambda_2 \frac{1}{a(a+1)} g^2 + \cdots\right]$, $0 < g$

[1] The pair for the oscillation cis $(-2\pi ft)$ and the unit impulse is $F(f)$, $G(-g)$ which differs from the tabulated pair only in the sign of the epoch; similarly, for the oscillation cos $(2\pi ft)$ or sin $(2\pi ft)$ the only change is the substitution for $G(g)$ of the even part of $G(g)$ or of the odd part of $-iG(g)$, the pairs being $F(f)$, $\frac{1}{2}[G(g) + G(-g)]$, or $F(f)$, $-i\frac{1}{2}[G(g) - G(-g)]$, respectively. Every pair in Table I may be thrown into the form of an evaluated Fourier integral by equating the pair after writing either $\int_{-\infty}^{\infty} df$ cis $(2\pi fg)$ before the coefficient $F(f)$ or $\int_{-\infty}^{\infty} dg$ cis $(-2\pi fg)$ before the coefficient $G(g)$. Every pair in Table I may also be regarded as an operational expression $F(p/i2\pi)$ of the operator $p = i2\pi f = d/dg$ with $G(g)$ its explicit expression in g.

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
109.*	$\frac{1}{p^a}(\lambda_0 + \lambda_1 p + \lambda_2 p^2 + \cdots)$, $0 < a < 1$, lim by 501*	$\frac{1}{\Gamma(a)g^{1-a}}\left[\lambda_0 - \lambda_1(1-a)\frac{1}{g} + \lambda_2(1-a)(2-a)\frac{1}{g^2} + \cdots\right]$, $0 < g$
110.*	$\frac{1}{\sqrt{p}}\left(\lambda_0 + \lambda_1\frac{1}{p} + \lambda_2\frac{1}{p^2} + \lambda_3\frac{1}{p^3} + \cdots\right)$, lim by 518*	$\frac{1}{\sqrt{\pi g}}\left[\lambda_0 + \lambda_1\frac{2g}{1} + \lambda_2\frac{(2g)^2}{1\cdot 3} + \lambda_3\frac{(2g)^3}{1\cdot 3\cdot 5} + \cdots\right]$, $0 < g$
111.*	$\frac{1}{\sqrt{p}}(\lambda_0 + \lambda_1 p + \lambda_2 p^2 + \lambda_3 p^3 + \cdots)$, lim by 502*	$\frac{1}{\sqrt{\pi g}}\left[\lambda_0 - \lambda_1\frac{1}{2g} + \lambda_2\frac{1\cdot 3}{(2g)^2} - \lambda_3\frac{1\cdot 3\cdot 5}{(2g)^3} + \cdots\right]$, $0 < g$
112.	$\lambda_0\phi_0(f) + \lambda_1\phi_1(f) + \lambda_2\phi_2(f) + \cdots$	$\lambda_0\phi_0(g) + i\lambda_1\phi_1(g) + i^2\lambda_2\phi_2(g) + \cdots$

Part 2. Elementary Combinations and Transformations

201.	$F_1 \pm F_2$	$G_1 \pm G_2$
202.[2]	$F_1 F_2$	$\int_{-\infty}^{\infty} G_1(x)G_2(g-x)dx$
203.	$\int_{-\infty}^{\infty} F_1(-x)F_2(f+x)dx$	$G_1 G_2$
204.	λF	λG

[2] From (202) or (203), with g (or f) = 0, and (215) and (217) follow the important identities for the integrated product of two pairs of coefficients and for the integrated squared moduli of a pair of coefficients:

$$\int_{-\infty}^{\infty} F_1(f)F_2(\pm f)df = \int_{-\infty}^{\infty} G_1(g)G_2(\mp g)dg,$$

$$\int_{-\infty}^{\infty} |F^2|df = \int_{-\infty}^{\infty} |G^2|dg,$$

$$\int_{-\infty}^{\infty} F_1(x)G_2(x)dx = \int_{-\infty}^{\infty} G_1(x)F_2(x)dx.$$

The symmetry of these identities is to be noted; this would not be the case if the radian frequency $2\pi f$ were employed in place of the cyclic frequency f.

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
205.	$F(af)$	$\frac{1}{a} G\left(\frac{g}{a}\right)$
206.	$F(f - f_0) = F\left(\frac{p - p_0}{i2\pi}\right)$	$\text{cis}(2\pi f_0 g) G = e^{p_0 g} G$
207.	$\text{cis}(-2\pi f g_0) F = e^{-p g_0} F$	$G(g - g_0)$
208.	pF	$D_g G$
209.	$D_p F = \frac{1}{i2\pi} D_f F$	$-gG$
210.	$\frac{1}{p} F$	$\int_{-\infty}^{g} G dg = D_g^{-1} G$
211.	$\int_{-i\infty}^{p} F dp = i2\pi \int_{-\infty}^{f} F df = D_p^{-1} F$	$-\frac{1}{g} G$
212.	$D_\lambda F$	$D_\lambda G$
213.	$\int_{\lambda_0}^{\lambda} F d\lambda = D_\lambda^{-1} F$	$\int_{\lambda_0}^{\lambda} G d\lambda = D_\lambda^{-1} G$
214.	$F(-f)$	$G(-g)$
215.	$\bar{F}(\pm f)$	$\bar{G}(\mp g)$
216.	$F(f) \pm \bar{F}(f)$	$G(g) \pm \bar{G}(-g)$
217.	$G(\pm f)$	$F(\mp g)$
218.	$G(\pm ip)$	$\frac{1}{2\pi} F\left(\frac{\pm g}{2\pi}\right)$
219.	$\frac{F(f)}{p - p_0}$	$e^{p_0 g} \int_{-\infty}^{g} e^{-p_0 g} G(g) dg$
220.	$\frac{p}{p - p_0} F(f)$	$G(g) + p_0 e^{p_0 g} \int_{-\infty}^{g} e^{-p_0 g} G(g) dg$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
221.	$f^{\pm\frac{1}{2}}D_f^{\pm1}(f^{\pm\frac{1}{2}}F) = (\frac{1}{2} + fD_f)^{\pm1}F$	$-\, g^{\pm\frac{1}{2}}D_g^{\pm1}(g^{\pm\frac{1}{2}}G) = -\,(\frac{1}{2} + gD_g)^{\pm1}G$
222.	$e^{\pm\pi f^2}D_f^{\nu}(e^{\mp\pi f^2}F) = (\mp 2\pi f + D_f)^{\nu}F$	$i^{\pm\nu}e^{\pm\pi g^2}D_g^{\nu}(e^{\mp\pi g^2}G) = i^{\pm\nu}(\mp 2\pi g + D_g)^{\nu}G$
223.[3]	$F_n(f), \quad n = 0, 1, 2, \cdots, 11$ where $F_n(f) = \frac{1}{4}[F(f) + i^{2n}F(-f) + i^{-n}G(f) + i^nG(-f)]$, $F_{n+4} = R(F_n), \quad F_{n+8} = I(F_n), \quad n = 0, 1, 2, 3$	$i^nF_n(g)$
224.	$F_0(f) + F_1(f) + F_2(f) + F_3(f)$ where F_n is as for 223.	$F_0(g) + iF_1(g) - F_2(g) - iF_3(g)$
225.	$F_4(f) + F_6(f) + F_5(f) + F_7(f)$ $+ i[F_8(f) + F_{10}(f) + F_9(f) + F_{11}(f)]$ where F_n is as for 223.	$F_4(g) - F_6(g) - F_9(g) + F_{11}(g)$ $+ i[F_8(g) - F_{10}(g) + F_5(g) - F_7(g)]$.

Part 3. Key Pairs

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse	
301.	$\sec p$; $\left[\gamma(p + \lambda) \text{ for } p, \text{ if } -\frac{1}{2}\pi R(1/\gamma) < R(\lambda) < \frac{1}{2}\pi R(1/\gamma)\right]$	$\frac{1}{2}\,\mathrm{sech}(\frac{1}{2}\pi g)$	
302.	$p^{\frac{1}{2}(\alpha-1)}K_{\alpha-1}(\sqrt{p})$; $\left[\gamma(p + \rho) \text{ for } p, \text{ and } \nu \text{ for } \alpha \text{ with } (p + \beta) \text{ for } p\right]$	$(2g)^{-\alpha}\exp\left(-\dfrac{1}{4g}\right)$,	$0 < g$
303.	$\dfrac{1}{p}\exp\left(-\dfrac{a^2+1}{p}\right)I_{\alpha-1}\left(\dfrac{2a}{p}\right)$; $\left[\gamma(p + \beta) \text{ for } p, \text{ and } \sqrt{\gamma\delta} \text{ for } a \text{ with } (p + \beta) \text{ for } p\right]$	$J_{\alpha-1}(2a\sqrt{g})J_{\alpha-1}(2\sqrt{g})$,	$0 < g$
304.	$\exp(\frac{1}{4}p^2)D_{-\alpha}(p)$; $[\sqrt{\gamma}(p + \rho) \text{ for } p]$	$\dfrac{1}{\Gamma(\alpha)}g^{\alpha-1}\exp(-\frac{1}{2}g^2)$,	$0 < g$

The coefficients of the i^n-multiple pairs satisfy the following integral equations:

$$F_n(f) = (-1)^{\frac{1}{2}}\; 2\int_0^\infty F_n(g)\cos(2\pi fg)dg, \quad n = 0, 2, 4, 6, 8, 10$$

$$F_n(f) = (-1)^{\frac{1}{2}(n-1)}2\int_0^\infty F_n(g)\sin(2\pi fg)dg, \quad n = 1, 3, 5, 7, 9, 11$$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
305.	$p^{-\frac{1}{2}\alpha}\exp\left(\frac{1}{4p}\right)D_{-\alpha}\left(\frac{1}{\sqrt{p}}\right)$; [$\gamma(p+\rho)$ for p]	$\frac{1}{\Gamma(\alpha)}(2g)^{\frac{1}{2}\alpha-1}\exp(-\sqrt{2g})$, $0 < g$
306.	$(p^2+b^2)^{\frac{1}{4}-\frac{1}{2}\alpha}K_{\alpha-\frac{1}{2}}(\sqrt{p^2+b^2})$, $0 < R(\alpha) < \frac{3}{2}$; [($p+\rho$) for p, and δ for b without the restriction on α with ($p+\beta$) for p, if $-R(\beta) < R(i\delta) < R(\beta)$]	$\sqrt{\frac{\pi}{2}}\left(\frac{g^2-1}{b^2}\right)^{\frac{1}{2}\alpha-\frac{1}{4}}J_{\alpha-1}(b\sqrt{g^2-1})$, $1<g$
307.	$(\delta^2-p^2)^{\frac{1}{4}-\frac{1}{2}\alpha}J_{\alpha-\frac{1}{2}}(\sqrt{\delta^2-p^2})$; [($p+\lambda$) for p]	$\frac{1}{\sqrt{2\pi}}\left(\frac{1-g^2}{\delta^2}\right)^{\frac{1}{2}\alpha-\frac{1}{4}}J_{\alpha-1}(\delta\sqrt{1-g^2})$, $-1 < g < 1$

Part 4. Rational Algebraic Functions of f.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
401.*	$p^n = \lim_{a\to 0} p^n e^{-\pi a^2 f^2}$	$\mathfrak{S}_n(g) = \lim_{a\to 0}\frac{1}{a}D_g^n e^{-\pi a^{-2}g^2}$
402.*	$p^n = \lim_{a\to 0}\sum_{k=1}^{n+1}\frac{(-1)^{k-1}(n+1)!(1-e^{-kap})}{k!(n-k+1)!\,a^{n+1}p}$	$\mathfrak{S}_n(g)$
403.*	$1 = \lim_{\beta\to 0} e^{-\beta\lvert p-p_0\rvert}$	$\mathfrak{S}_0(g)$, unit impulse at $g = 0$
404.*	$p = \lim_{\beta\to 0} pe^{-\pi\beta f^2}$	$\mathfrak{S}_1(g)$, negative unit doublet at $g = 0$
405.*	$\lvert p^{2n}\rvert = \lim_{\beta\to 0} D_\beta^{2n}e^{-\beta\lvert p\rvert}$	$(-1)^n\mathfrak{S}_{2n}(g)$
406.*	$\lvert p^{2n+1}\rvert = -\lim_{\beta\to 0} D_\beta^{2n+1}e^{-\beta\lvert p\rvert}$	$\frac{(-1)^{n+1}(2n+1)!}{\pi g^{2n+2}}$
407.*	$\lvert p\rvert$, lim by 406*	$-\frac{1}{\pi g^2}$
408.*	$p^{-n} = \lim_{\beta\to 0}(p+\beta)^{-n}$, $0 < n$	$\frac{g^{n-1}}{(n-1)!}$, $0 < g$
409.*	$p^{-n} = \lim_{\beta\to 0}(p-\beta)^{-n}$, $0 < n$	$\frac{-g^{n-1}}{(n-1)!}$, $g < 0$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
410.*	$p^{-n} = \lim_{a \to 0} p^{-n} e^{-\pi a^2 f^2}, \quad 0 < n$	$\mathfrak{S}_{-n}(g) = \lim_{a \to 0} \frac{1}{a} D_g^{-n} e^{-\pi a^{-2} g^2}$ $= \left(\lambda_1 \pm \frac{1}{2(n-1)!} \right) g^{n-1} + \lambda_2 g^{n-2} + \lambda_3 g^{n-3} + \cdots + \lambda_n, \quad 0 < \pm g$
411.*	$p^{-n} = \lim_{\beta \to 0} \left[\frac{\frac{1}{2}}{(p-\beta)^n} + \frac{\frac{1}{2}}{(p+\beta)^n} + \sum_{k=1}^{n} \left(\frac{\lambda_k (n-k)!}{(p+\beta)^{n-k+1}} - \frac{\lambda_k (n-k)!}{(p-\beta)^{n-k+1}} \right) \right], \quad 0 < n$	$\mathfrak{S}_{-n}(g)$
415.*	$\frac{1}{p} = \lim_{\beta \to 0} \left(\frac{\frac{1}{2} - \lambda}{p - \beta} + \frac{\frac{1}{2} + \lambda}{p + \beta} \right)$	$\mathfrak{S}_{-1}(g) = \lambda \pm \frac{1}{2}$, $0 < \pm g$, unit step at $g = 0$
416.*	$\frac{1}{p^2} = \lim_{\beta \to 0} \left[\frac{\frac{1}{2} - \lambda}{(p-\beta)^2} + \frac{\frac{1}{2} + \lambda}{(p+\beta)^2} - \frac{2\beta\mu}{p^2 - \beta^2} \right]$	$\mathfrak{S}_{-2}(g) = \frac{1}{2} \lvert g \rvert + \lambda g + \mu$
421.*	$F(f) = \lim_{a \to 0} F_1(f)$, where F is any proper rational fraction in p with n distinct poles, the degree of pole z_j being n_j. All pure imaginary poles in F are the limits of corresponding poles in F_1 which have assigned real parts $\pm a$.	$\sum_{j=1}^{n} \sum_{k=1}^{n} \pm \lambda_{jk} e^{z_j g} g^{n_j - k}$, $0 < \pm g, \pm R(z_j) < 0$, where $\lambda_{jk} = \frac{\{D_p^{k-1}[F(f)(p - z_j)^{n_j}]\}_{p=z}}{(k-1)!(n_j - k)!}$ and the upper or lower signs for each term are employed according as the real part of z_j (either actual or vestigial) is less than or greater than zero.
431.	$\frac{1}{(p+\beta)^n}, \quad 0 < n$	$\frac{1}{(n-1)!} e^{-\beta g} g^{n-1}, \quad 0 < g$
432.	$\frac{1}{(p-\beta)^n}, \quad 0 < n$	$\frac{-1}{(n-1)!} e^{\beta g} g^{n-1}, \quad g < 0$
433.	$\frac{1}{(p^2 - \beta^2)^n}, \quad 0 < n$	$\frac{(-1)^n}{(n-1)!} e^{-\beta \lvert g \rvert} \sum_{k=1}^{n} \frac{(n+k-2)! \lvert g^{n-k} \rvert}{(k-1)!(n-k)!(2\beta)^{n+k-1}}$ $= \frac{(-1)^n \lvert g^{n-\frac{1}{2}} \rvert K_{n-\frac{1}{2}}(\beta \lvert g \rvert)}{(n-1)! \, \pi^{\frac{1}{2}} (2\beta)^{n-\frac{1}{2}}}$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
438.	$\frac{1}{p+\beta}$		$e^{-\beta g}$,	$0 < g$
439.	$\frac{1}{p-\beta}$		$-e^{\beta g}$,	$g < 0$
440.*	$\frac{1}{p-p_0}$,	lim by 415*	$\text{cis}\,(2\pi f_0 g)\mathfrak{S}_{-1}(g)$	
441.*	$\frac{p}{p+\beta}$,	lim by 403*	$\mathfrak{S}_0(g) - \beta e^{-\beta g}$,	$0 < g$
442.	$\frac{1}{(p+\beta)^2}$		$ge^{-\beta g}$,	$0 < g$
443.	$\frac{1}{(p-\beta)^2}$		$-ge^{\beta g}$,	$g < 0$
444.	$\frac{1}{p^2-\beta^2}$		$-\frac{1}{2\beta}e^{-\beta\|g\|}$	
445.	$\frac{p}{p^2-\beta^2}$		$\pm\frac{1}{2}e^{\mp\beta g}$,	$0 < \pm g$
446.*	$\frac{a}{p^2+a^2}$,	lim by 415*	$\sin ag\ \mathfrak{S}_{-1}(g)$	
447.*	$\frac{p}{p^2+a^2}$,	lim by 415*	$\cos ag\ \mathfrak{S}_{-1}(g)$	
448.	$\frac{1}{(p+\alpha)(p+\beta)}$		$\frac{e^{-\beta g} - e^{-\alpha g}}{\alpha-\beta}$,	$0 < g$
449.	$\frac{p}{(p+\alpha)(p+\beta)}$		$\frac{\alpha e^{-\alpha g} - \beta e^{-\beta g}}{\alpha-\beta}$,	$0 < g$
450.	$\frac{1}{(p+\beta)^3}$		$\frac{1}{2}g^2e^{-\beta g}$,	$0 < g$
451.	$\frac{1}{(p-\beta)^3}$		$-\frac{1}{2}g^2e^{\beta g}$,	$g < 0$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
452.	$\dfrac{1}{(p+\alpha)(p+\beta)(p+\gamma)}$		$\dfrac{(\gamma-\beta)e^{-\alpha g}+(\alpha-\gamma)e^{-\beta g}+(\beta-\alpha)e^{-\gamma g}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$,	$0 < g$
453.	$\dfrac{p}{(p+\alpha)(p+\beta)(p+\gamma)}$		$\dfrac{\alpha(\beta-\gamma)e^{-\alpha g}+\beta(\gamma-\alpha)e^{-\beta g}+\gamma(\alpha-\beta)e^{-\gamma g}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$,	$0 < g$
454.	$\dfrac{\alpha\beta}{p(p+\alpha)(p+\beta)}-\dfrac{1}{p}$		$\dfrac{\beta e^{-\alpha g}-\alpha e^{-\beta g}}{\alpha-\beta}$,	$0 < g$
455.*	$\dfrac{a^2}{p(p^2+a^2)}$,	lim by 415*	$2\sin^2(\tfrac{1}{2}ag)\mathfrak{S}_{-1}(g)$	
456.	$\dfrac{1}{(p+b+ia)(p+b-ia)}$		$\dfrac{1}{a}\sin ag\, e^{-bg}$,	$0 < g$
457.	$\dfrac{1}{(p-b+ia)(p-b-ia)}$		$-\dfrac{1}{a}\sin ag\, e^{bg}$,	$g < 0$
458.	$\dfrac{p}{(p+b+ia)(p+b-ia)}$		$\left(\cos ag-\dfrac{b}{a}\sin ag\right)e^{-bg}$,	$0 < g$
459.	$\dfrac{p}{(p-b+ia)(p-b-ia)}$		$-\left(\cos ag+\dfrac{b}{a}\sin ag\right)e^{bg}$,	$g < 0$

Part 5. Irrational Algebraic Functions of f.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
501.*	$p^{n-\alpha}=\lim\limits_{\beta\to 0} p^{n-\alpha}e^{-\beta\lvert p\rvert}$,	$0 < R(\alpha) < 1$	$\dfrac{1}{\Gamma(\alpha-n)}g^{\alpha-n-1}$,	$0 < g$
502.*	$p^{n-\frac{1}{2}}=\lim\limits_{\beta\to 0} p^{n-\frac{1}{2}}e^{-\beta\lvert p\rvert}$,	$0 < n$	$\dfrac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{(\frac{1}{2}\pi)^{\frac{1}{2}}}(2g)^{-n-\frac{1}{2}}$,	$0 < g$
503.*	$p^{\frac{1}{2}}$,	lim by 502*	$-\tfrac{1}{2}\pi^{-\frac{1}{2}}g^{-\frac{3}{2}}$,	$0 < g$
504.*	$p^{\frac{3}{2}}$,	lim by 502*	$\tfrac{3}{4}\pi^{-\frac{1}{2}}g^{-\frac{5}{2}}$,	$0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
505.*	$\|f^{\frac{1}{2}}\|$,	lim by 503*	$\frac{-1}{4\pi}\|g^{-\frac{3}{2}}\|$	
506.*	$(p+\beta)^{\frac{1}{2}}$,	lim by 820	$-\frac{1}{2}\pi^{-\frac{1}{2}}e^{-\beta g}g^{-\frac{3}{2}}$,	$0 < g$
516.*	$p^{-\alpha} = \lim_{\beta\to 0}(p+\beta)^{-\alpha}$,	br 0	$\frac{1}{\Gamma(\alpha)}g^{\alpha-1}$,	br 0, $0 < g$
517.*	$p^{-\alpha} = \lim_{\beta\to 0}(p-\beta)^{-\alpha}$,	br $(-\frac{1}{2})$	$\frac{-1}{\Gamma(\alpha)}g^{\alpha-1}$,	br 0, $g < 0$
518.*	$p^{-n-\frac{1}{2}} = \lim_{\beta\to 0}(p+\beta)^{-n-\frac{1}{2}}$,	br 0, $0 < n$	$\frac{(\frac{1}{2}\pi)^{-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)}(2g)^{n-\frac{1}{2}}$,	br 0, $0 < g$
519.*	$p^{-n-\frac{1}{2}} = \lim_{\beta\to 0}(p-\beta)^{-n-\frac{1}{2}}$,	br $\frac{1}{2}$, $0 < n$	$\frac{(\frac{1}{2}\pi)^{-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)}(2g)^{n-\frac{1}{2}}$,	br 0, $g < 0$
520.*	$p^{-\frac{3}{2}}$,	lim by 518*	$2(g/\pi)^{\frac{1}{2}}$,	$0 < g$
521.	$p^{-\alpha}$,	$0 < R(\alpha) < 1$	$\frac{1}{\Gamma(\alpha)}g^{\alpha-1}$,	$0 < g$
522.	$p^{-\frac{1}{2}}$		$(\pi g)^{-\frac{1}{2}}$,	$0 < g$
523.	$\|f^{-\frac{1}{2}}\|$		$\|g^{-\frac{1}{2}}\|$	
524.	$(p+\beta)^{-\alpha}$,	br v	$\frac{1}{\Gamma(\alpha)}e^{-i2\pi\alpha(v+w)}e^{-\beta g}g^{\alpha-1}$,	br w, $0 < g$
525.	$(p-\beta)^{-\alpha}$,	br $(v-\frac{1}{2})$	$\frac{-1}{\Gamma(\alpha)}e^{-i2\pi\alpha(v+w)}e^{\beta g}g^{\alpha-1}$,	br w, $g < 0$
526.	$(p+\beta)^{-\frac{1}{2}}$,	br 0	$e^{-\beta g}(\pi g)^{-\frac{1}{2}}$,	br 0, $0 < g$
527.	$(p-\beta)^{-\frac{1}{2}}$,	br $\frac{1}{2}$	$e^{\beta g}(\pi g)^{-\frac{1}{2}}$,	br 0, $g < 0$
528.	$(p-\beta)^{-\frac{1}{2}}$,	br 0	$-e^{\beta g}(\pi g)^{-\frac{1}{2}}(\text{erf}\sqrt{\beta g} - \frac{1}{2} \mp \frac{1}{2})$,	br 0, $0 < \pm g$
529.	$(p+\beta)^{-\frac{3}{2}}$,	br 0	$2e^{-\quad g}(g/\pi)^{\frac{1}{2}}$,	br 0, $0 < g$
530.	$(p+\beta)^{1-\alpha} - (p+\gamma)^{1-\alpha}$		$\frac{1}{\Gamma(\alpha-1)}g^{\alpha-2}(e^{-\beta g} - e^{-\gamma g})$,	$0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
541.	$\frac{\sqrt{p}}{p+\gamma}$	$\frac{1}{\sqrt{\pi g}} + \sqrt{-\gamma}\, e^{-\gamma g} \operatorname{erf} \sqrt{-\gamma g}, \quad 0 < g$
542.	$\frac{1}{(p+\gamma)\sqrt{p}}$	$\frac{1}{\sqrt{-\gamma}} e^{-\gamma g} \operatorname{erf} \sqrt{-\gamma g}, \quad 0 < g$
543.	$\frac{1}{1+\sqrt{\beta p}}$	$\frac{1}{\sqrt{\pi\beta g}} - \frac{1}{\beta} \exp \frac{g}{\beta} \operatorname{erfc} \sqrt{\frac{g}{\beta}}, \quad 0 < g$
544.*	$\frac{p}{1+\sqrt{\beta p}}$	$-\frac{1}{\beta} \mathfrak{S}_0(g) + \frac{2g-\beta}{2\beta g\sqrt{\pi\beta g}} - \frac{1}{\beta^2} \exp \frac{g}{\beta} \operatorname{erfc} \sqrt{\frac{g}{\beta}}, \quad 0 < g$
545.	$\frac{p}{(p+\gamma)(1+\sqrt{\beta p})}$	$\frac{-\gamma e^{-\gamma g}}{1+\beta\gamma}(1 - \sqrt{-\beta\gamma} \operatorname{erf} \sqrt{-\gamma g}) + \frac{1}{\sqrt{\pi\beta g}} - \frac{\exp (g/\beta)}{\beta(1+\beta\gamma)} \operatorname{erfc} \sqrt{\frac{g}{\beta}}, \quad 0 < g$
546.	$\frac{1}{(p+\gamma)\sqrt{p+\beta}}$	$\frac{1}{\sqrt{\beta-\gamma}} e^{-\gamma g} \operatorname{erf} \sqrt{(\beta-\gamma)g}, \quad 0 < g$
547.	$\frac{\sqrt{\beta}}{p\sqrt{p+\beta}} - \frac{1}{p}$	$-\operatorname{erfc} \sqrt{\beta g}, \quad 0 < g$
548.	$\frac{1}{p}\sqrt{\frac{p}{\beta}+1} - \frac{1}{p}$	$\frac{1}{\sqrt{\pi\beta g}} e^{-\beta g} - \operatorname{erfc} \sqrt{\beta g}, \quad 0 < g$
549.	$\frac{\sqrt{p+\beta}}{p+\gamma}$	$\frac{1}{\sqrt{\pi g}} e^{-\beta g} + \sqrt{\beta-\gamma}\, e^{-\gamma g} \operatorname{erf} \sqrt{(\beta-\gamma)g}, \quad 0 < g$
550.*	$\frac{\sqrt{p}}{1+\sqrt{\beta p}}$	$\frac{1}{\sqrt{\beta}} \mathfrak{S}_0(g) - \frac{1}{\beta\sqrt{\pi g}} + \frac{1}{\beta\sqrt{\beta}} \exp \frac{g}{\beta} \operatorname{erfc} \sqrt{\frac{g}{\rho}}, \quad 0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
551.	$\dfrac{1}{\sqrt{p}(1+\sqrt{\beta p})}$	$\dfrac{1}{\sqrt{\beta}} \exp \dfrac{g}{\beta} \operatorname{erfc} \sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
552.	$\dfrac{\sqrt{p}}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{\sqrt{-\gamma}\, e^{-\gamma g}}{1+\beta\gamma}(\operatorname{erf}\sqrt{-\gamma g} - \sqrt{-\beta\gamma}) + \dfrac{\exp(g/\beta)}{\sqrt{\beta}(1+\beta\gamma)} \operatorname{erfc}\sqrt{\dfrac{g}{\beta}}, \quad 0 < g$
553.*	$\sqrt{\dfrac{p+\alpha}{p}}$	$\mathfrak{S}_0(g) + \dfrac{\alpha}{2} e^{-\frac{1}{2}\alpha g}[I_1(\tfrac{1}{2}\alpha g) + I_0(\tfrac{1}{2}\alpha g)], \quad 0 < g$
554.*	$\dfrac{1}{p}\sqrt{\dfrac{p+\alpha}{p}}$	$e^{-\frac{1}{2}\alpha g}[\alpha g I_1(\tfrac{1}{2}\alpha g) + (1+\alpha g) I_0(\tfrac{1}{2}\alpha g)], \quad 0 < g$
555.	$\dfrac{1}{\sqrt{(p+\alpha)(p+\beta)}}$	$e^{-\frac{1}{2}(\alpha+\beta)g} I_0[\tfrac{1}{2}(\alpha-\beta)g], \quad 0 < g$
556.*	$\sqrt{p^2+a^2}$	$\mathfrak{S}_1(g) + \dfrac{a}{g} J_1(ag), \quad 0 < g$
557.	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(ag), \quad 0 < g$
558.	$\dfrac{1}{\sqrt{\beta^2 - p^2}}$	$\dfrac{1}{\pi} K_0(\beta \vert g \vert)$
559.*	$\dfrac{\sqrt{p+\alpha}}{\sqrt{p}+\sqrt{p+\alpha}}$	$\tfrac{1}{2}\mathfrak{S}_0(g) + \dfrac{1}{2g} e^{-\frac{1}{2}\alpha g} I_1(\tfrac{1}{2}\alpha g), \quad 0 < g$
560.	$\dfrac{\sqrt{p+\alpha}}{p(\sqrt{p}+\sqrt{p+\alpha})} - \dfrac{1}{p}$	$-\tfrac{1}{2} e^{-\frac{1}{2}\alpha g}[I_1(\tfrac{1}{2}\alpha g) + I_0(\tfrac{1}{2}\alpha g)], \quad 0 < g$
571.	$(p^2+b^2)^{-\alpha}, \quad 0 < R(\alpha) < 1$	$\dfrac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\dfrac{g}{2b}\right)^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(bg), \quad 0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse	
572.	$[(p+\beta)^2+r^2]^{-\alpha}$	$\frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{g}{2r}\right)^{\alpha-\frac{1}{2}} e^{-\beta g} J_{\alpha-\frac{1}{2}}(rg),$	$0 < g$
573.	$\sqrt{\frac{p}{p+\beta}}(\sqrt{p+\beta}+\sqrt{p})^{-2\alpha}$	$\frac{1}{4}\beta^{-\alpha+1}e^{-\frac{1}{2}\beta g}[I_{\alpha-1}(\frac{1}{2}\beta g)-2I_\alpha(\frac{1}{2}\beta g) + I_{\alpha+1}(\frac{1}{2}\beta g)],$	$0 < g$
574.	$\frac{(\sqrt{p+\beta}+\sqrt{p})^{-2\sigma}}{\sqrt{p(p+\beta)}}$	$\beta^{-\sigma}e^{-\frac{1}{2}\beta g}I_\sigma(\frac{1}{2}\beta g),$	$0 < g$
575.	$\frac{[\sqrt{(p+\rho)^2+a^2}+(p+\rho)]^{-\alpha+1}}{\sqrt{(p+\rho)^2+a^2}}$	$a^{-\alpha+1}e^{-\rho g}J_{\alpha-1}(ag),$	$0 < g$
576.	$[\sqrt{(p+\rho)^2+a^2}+(p+\rho)]^{-\alpha}$	$\frac{\alpha e^{-\rho g}}{a^\alpha g} J_\alpha(ag),$	$0 < g$

Part 6. Exponential and Trigonometric Functions of f or f^{-1}.

601.*	e^{-rp}	$\mathfrak{S}_0(g-r)$	
602.*	$\frac{e^{-rp}}{p}$	$\mathfrak{S}_{-1}(g-r)$	
603.	$\frac{1}{p}(1-e^{-ap})$	$1,$	$0 < g < a$
604.	$\frac{e^{-rp}}{p+\beta}$	$e^{-\beta(g-r)},$	$r < g$
611.	$\frac{\alpha}{\sin \alpha p}-\frac{1}{p}$	$\frac{1}{2}\tanh\frac{\pi g}{2\alpha} \mp \frac{1}{2},$	$0 < \pm g$
612.*	$\tan \alpha p$	$\frac{-1}{2\alpha \sinh \frac{\pi g}{2\alpha}}$	
613.*	$\alpha \operatorname{ctn} \alpha p - \frac{1}{p}$	$\frac{1}{2}\operatorname{ctnh}\frac{\pi g}{2\alpha} \mp \frac{1}{2},$	$0 < \pm g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
614.	$\dfrac{p}{\sin \alpha p}$	$\dfrac{\pi}{4\alpha^2 \cosh^2 \dfrac{\pi g}{2\alpha}}$
615.	$\dfrac{\sin \beta p}{\sin \alpha p}, \quad R(\beta) < R(\alpha)$	$\dfrac{\dfrac{1}{2\alpha}\sin\dfrac{\pi\beta}{\alpha}}{\cosh\dfrac{\pi g}{\alpha} + \cos\dfrac{\pi\beta}{\alpha}}$
616.	$\dfrac{\cos \beta p}{\cos \alpha p}, \quad R(\beta) < R(\alpha)$	$\dfrac{\dfrac{1}{\alpha}\cos\dfrac{\pi\beta}{2\alpha}\cosh\dfrac{\pi g}{2\alpha}}{\cosh\dfrac{\pi g}{\alpha} + \cos\dfrac{\pi\beta}{\alpha}}$
617.	$\dfrac{\alpha \sin \beta p}{\beta p \sin \alpha p} - \dfrac{1}{p}, \quad R(\beta) < R(\alpha)$	$\dfrac{\alpha}{\pi\beta}\tan^{-1}\dfrac{\tanh\dfrac{\pi g}{2\alpha}}{\operatorname{ctn}\dfrac{\pi\beta}{2\alpha}} \mp \dfrac{1}{2}, \quad 0 < \pm g$
618.	$\dfrac{\cos \beta p}{p \cos \alpha p} - \dfrac{1}{p}, \quad R(\beta) < R(\alpha)$	$-\dfrac{1}{\pi}\tan^{-1}\dfrac{\cos\dfrac{\pi\beta}{2\alpha}}{\sinh\dfrac{\pi g}{2\alpha}}$
619.*	$\cosh(ap)$	$\frac{1}{2}[\mathfrak{S}_0(g+a) + \mathfrak{S}_0(g-a)]$
620.	$\dfrac{\cosh(ap)}{p} - \dfrac{1}{p}$	$\mp \frac{1}{2}, \quad 0 < \pm g < a$
621.	$\dfrac{\cosh(ap)}{(p-p_0)\cosh(ap_0)} - \dfrac{1}{p-p_0}$	$\frac{1}{2}\operatorname{cis}(2\pi f_0 g)[\tanh(ap_0) \mp 1], \quad 0 < \pm g < a$
622.	$\dfrac{\sinh(ap)}{p}$	$\frac{1}{2}, \quad -a < g < a$
623.	$\dfrac{\sinh(ap)}{ap^2} - \dfrac{1}{p}$	$\dfrac{g}{2a} \mp \frac{1}{2}, \quad 0 < \pm g < a$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
624.	$\frac{p_0 \sinh(ap)}{p(p-p_0)\sinh(ap_0)} - \frac{1}{p-p_0}$	$\frac{1}{2}\{\text{cis}(2\pi f_0 g)[\text{ctnh}(ap_0) \mp 1] - \text{csch}(ap_0)\}$, $0 < \pm g < a$
625.	$\text{sech}\ \pi f$	$\text{sech}\ \pi g$
631.	$p^{\alpha-1}e^{-\beta\lvert p\rvert}$	$\frac{\Gamma(\alpha)}{i2\pi}[(-g-i\beta)^{-\alpha} - (-g+i\beta)^{-\alpha}]$
632.	$e^{-\beta\lvert p\rvert}$	$\frac{1}{\pi}\cdot\frac{\beta}{\beta^2+g^2}$
633.	$\frac{1}{p}e^{-\beta\lvert p\rvert} - \frac{1}{p}$	$-\frac{1}{\pi}\tan^{-1}\frac{\beta}{g}$
634.	$\frac{e^{-\beta\lvert p\rvert}}{\sqrt{\lvert p\rvert}}$	$\sqrt{\frac{\beta+\sqrt{\beta^2+g^2}}{2\pi(\beta^2+g^2)}}$
635.	$\lvert p\rvert e^{-\beta\lvert p\rvert}$	$\frac{\beta^2-g^2}{\pi(\beta^2+g^2)^2}$
645.*	$\sin(a\lvert p\rvert)$	$\frac{a}{\pi(a^2-g^2)}$
651.	$\frac{1}{\sqrt{p+\beta}}\exp\frac{\rho}{p+\beta}$	$\frac{e^{-\beta g}}{\sqrt{\pi g}}\cosh(2\sqrt{\rho g})$, $0 < g$
652.*	$\frac{1}{\sqrt{p}}\exp\frac{1}{4p}$	$\frac{1}{\sqrt{\pi g}}\cosh\sqrt{g}$, $0 < g$
653.	$(p+\beta)^{-\frac{3}{2}}\exp\frac{\rho}{p+\beta}$	$\frac{e^{-\beta g}}{\sqrt{\pi\rho}}\sinh(2\sqrt{\rho g})$, $0 < g$
654.*	$\exp\left(-\frac{1}{p}\right)$	$\mathfrak{S}_0(g) - \frac{1}{\sqrt{g}}J_1(2\sqrt{g})$, $0 < g$
655.	$\frac{1}{p}\exp\left(-\frac{1}{p}\right)$	$J_0(2\sqrt{g})$, $0 < g$
656.*	$\frac{1}{p^2}\exp\left(-\frac{1}{p}\right)$	$\sqrt{g}\,J_1(2\sqrt{g})$, $0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Part 7. Exponential and Trigonometric Functions of f^2.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
701.	$e^{\rho p^2}$, $\quad 0 < \lvert\rho\rvert$	$\frac{1}{2\sqrt{\pi\rho}} e^{-\frac{1}{4}g^2/\rho}$
702.	$\phi_n(f) = e^{\pi f^2} D_f{}^n e^{-2\pi f^2}$	$i^n\phi_n(g)$
703.	$\phi_0(f) = e^{-\frac{1}{2}x^2} = e^{-\pi f^2}$, $\quad x = f\sqrt{4\pi}$ in 703–711	$\phi_0(g) = e^{-\pi g^2}$
704.	$\phi_1(f) = -e^{-\frac{1}{2}x^2}(4\pi)^{\frac{1}{2}}x$	$i\phi_1(g)$
705.	$\phi_2(f) = e^{-\frac{1}{2}x^2}(4\pi)(x^2 - 1)$	$-\phi_2(g)$
706.	$\phi_3(f) = -e^{-\frac{1}{2}x^2}(4\pi)^{\frac{3}{2}}(x^3 - 3x)$	$-i\phi_3(g)$
707.	$\phi_4(f) = e^{-\frac{1}{2}x^2}(4\pi)^2(x^4 - 6x^2 + 3)$	$\phi_4(g)$
708.	$\phi_5(f) = -e^{-\frac{1}{2}x^2}(4\pi)^{\frac{5}{2}}(x^5 - 10x^3 + 15x)$	$i\phi_5(g)$
709.	$\phi_6(f) = e^{-\frac{1}{2}x^2}(4\pi)^3(x^6 - 15x^4 + 45x^2 - 15)$	$-\phi_6(g)$
710.	$\phi_7(f) = -e^{-\frac{1}{2}x^2}(4\pi)^{\frac{7}{2}}(x^7 - 21x^5 + 105x^3 - 105x)$	$-i\phi_7(g)$
711.	$\phi_8(f) = e^{-\frac{1}{2}x^2}(4\pi)^4(x^8 - 28x^6 + 210x^4 - 420x^2 + 105)$	$\phi_8(g)$
712.	$e^{-\pi f^2}(4\pi f^2 - 3)^2$	$e^{-\pi g^2}(4\pi g^2 - 3)^2$
713.	$p^n e^{-\pi\beta f^2}$	$\frac{1}{\sqrt{\beta}} D_g{}^n e^{-\pi g^2/\beta} = \frac{1}{\sqrt{\beta}(\sqrt{2\beta})^n} \times e^{-\frac{1}{2}\pi g^2/\beta}\phi_n\left(\frac{g}{\sqrt{2\beta}}\right)$
714.	$e^{-\pi\beta f^2}$	$\frac{1}{\sqrt{\beta}} e^{-\pi g^2/\beta}$
715.	$pe^{-\pi\beta f^2}$	$-\frac{2\pi g}{\beta\sqrt{\beta}} e^{-\pi g^2/\beta}$
716.	$p^2 e^{-\pi\beta f^2}$	$\frac{2\pi}{\beta^2\sqrt{\beta}} e^{-\pi g^2/\beta}(2\pi g^2 - \beta)$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
717.	$p^3e^{-\pi\beta f^2}$	$-\dfrac{4\pi^2 g}{\beta^3\sqrt{\beta}}e^{-\pi g^2/\beta}(2\pi g^2 - 3\beta)$
718.	$p^4e^{-\pi\beta f^2}$	$\dfrac{4\pi^2}{\beta^4\sqrt{\beta}}e^{-\pi g^2/\beta}(4\pi^2 g^4 - 12\pi\beta g^2 + 3\beta^2)$
719.	$p^n e^{-\frac{1}{2}\pi f^2}$	$\sqrt{2}D_g{}^n e^{-2\pi g^2} = \sqrt{2}e^{-\pi g^2}\phi_n(g)$
720.	$e^{-\frac{1}{2}\pi f^2}$	$\sqrt{2}e^{-2\pi g^2}$
721.	$pe^{-\frac{1}{2}\pi f^2}$	$-4\pi g\sqrt{2}e^{-2\pi g^2}$
722.	$p^2e^{-\frac{1}{2}\pi f^2}$	$4\pi\sqrt{2}e^{-2\pi g^2}(4\pi g^2 - 1)$
723.	$p^3e^{-\frac{1}{2}\pi f^2}$	$-16\pi^2 g\sqrt{2}e^{-2\pi g^2}(4\pi g^2 - 3)$
724.	$p^4e^{-\frac{1}{2}\pi f^2}$	$16\pi^2\sqrt{2}e^{-2\pi g^2}(16\pi^2 g^4 - 24\pi g^2 + 3)$
725.*	$\dfrac{1}{p}e^{-\pi\beta f^2}$	$\mathfrak{S}_{-1}(g) + \frac{1}{2}\operatorname{erf}(g\sqrt{\pi/\beta}) \mp \frac{1}{2}, \quad 0 < \pm g$
726.*	$\dfrac{1}{p^2}e^{-\pi\beta f^2}$	$\mathfrak{S}_{-2}(g) + \frac{1}{2}g\operatorname{erf}(g\sqrt{\pi/\beta})$ $+\dfrac{\sqrt{\beta}}{2\pi}e^{-\pi g^2/\beta} \mp \frac{1}{2}g, \quad 0 < \pm g$
727.	$\dfrac{1}{p}\exp(\alpha p^2) - \dfrac{1}{p}$	$\mp\frac{1}{2}\operatorname{erfc}\dfrac{\lvert g\rvert}{2\sqrt{\alpha}}, \quad 0 < \pm g$
728.	$\dfrac{1}{p - p_0}\exp[\alpha(p^2 - p_0^2)] - \dfrac{1}{p - p_0}$	$\frac{1}{2}\operatorname{cis}(2\pi f_0 g)\left(\operatorname{erf}\dfrac{g + 2\alpha p_0}{2\sqrt{\alpha}} \mp 1\right),$ $0 < \pm g$
729.	$\exp(\rho p^2 + \sigma p), \quad 0 < \lvert\rho\rvert$	$\dfrac{1}{2\sqrt{\pi\rho}}\exp\left[-\dfrac{(g + \sigma)^2}{4\rho}\right]$
751.	$\sin(ap^2)$	$\dfrac{1}{2\sqrt{\pi a}}\sin\left(\dfrac{g^2}{4a} - \dfrac{\pi}{4}\right)$
752.	$\cos(ap^2)$	$\dfrac{1}{2\sqrt{\pi a}}\sin\left(\dfrac{g^2}{4a} + \dfrac{\pi}{4}\right)$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
753.	$\frac{\sin(ap^2)}{p}$	$\frac{1}{2}\left[S\left(\frac{g}{\sqrt{2\pi a}}\right)-C\left(\frac{g}{\sqrt{2\pi a}}\right)\right]$
754.	$\frac{\cos(ap^2)}{p}-\frac{1}{p}$	$\frac{1}{2}\left[S\left(\frac{g}{\sqrt{2\pi a}}\right)+C\left(\frac{g}{\sqrt{2\pi a}}\right)\mp 1\right],\quad 0<\pm g$
755.	$\frac{\cos(ap^2)}{(p-p_0)\cos(ap_0^2)}-\frac{1}{p-p_0}$	$\frac{\operatorname{cis}(2\pi f_0 g)}{4\cos(ap_0^2)}\Big[\exp(iap_0^2)\operatorname{erf}\left(\frac{g}{2\sqrt{ia}}+p_0\sqrt{ia}\right)+\exp(-iap_0^2)\operatorname{erf}\left(\frac{g}{2\sqrt{-ia}}+p_0\sqrt{-ia}\right)\mp 2\cos(ap_0^2)\Big],\quad 0<\pm g$
756.	$\frac{\sin(ap^2)}{p^2}$	$\sqrt{\frac{a}{\pi}}\sin\left(\frac{g^2}{4a}+\frac{\pi}{4}\right)+\frac{g}{2}\left[S\left(\frac{g}{\sqrt{2\pi a}}\right)-C\left(\frac{g}{\sqrt{2\pi a}}\right)\right]$
757.	$\frac{\sin(ap^2)}{p^3}-\frac{a}{p}$	$\frac{1}{2}\Big[\left(a+\frac{g^2}{2}\right)S\left(\frac{g}{\sqrt{2\pi a}}\right)+\left(a-\frac{g^2}{2}\right)C\left(\frac{g}{\sqrt{2\pi a}}\right)+g\sqrt{\frac{a}{\pi}}\sin\left(\frac{g^2}{4a}+\frac{\pi}{4}\right)\mp a\Big],\quad 0<\pm g$
758.	$\sin(ap^2+\lambda)$	$\frac{1}{2\sqrt{\pi a}}\sin\left(\frac{g^2}{4a}+\lambda-\frac{\pi}{4}\right)$
759.	$\cos(ap^2+\lambda)$	$\frac{1}{2\sqrt{\pi a}}\sin\left(\frac{g^2}{4a}+\lambda+\frac{\pi}{4}\right)$
760.	$\operatorname{cis}[\pm\pi(f^2-\frac{1}{8})]$	$\operatorname{cis}[\mp\pi(g^2-\frac{1}{8})]$
761.	$\cos[\pi(f^2-\frac{1}{8})]$	$\cos[\pi(g^2-\frac{1}{8})]$
762.	$\sin[\pi(f^2-\frac{1}{8})]$	$-\sin[\pi(g^2-\frac{1}{8})]$

TABLE I (Continued)

Part 8. Other Elementary Transcendental Functions of f.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
801.	$\exp(-\alpha\sqrt{p})$	$\frac{\alpha}{2g\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right), \quad 0 < g$
802.	$p\exp(-\alpha\sqrt{p})$	$\left(\frac{\alpha^2}{2g}-3\right)\frac{\alpha}{4g^2\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right), \quad 0 < g$
803.	$\frac{1}{p}[1-\exp(-\alpha\sqrt{p})]$	$\text{erf}\,\frac{\alpha}{2\sqrt{g}}, \quad 0 < g$
804.*	$\frac{1}{p^2}[1-\exp(-\alpha\sqrt{p})]+\frac{\alpha^2}{2p}$	$\left(g+\frac{\alpha^2}{2}\right)\text{erf}\,\frac{\alpha}{2\sqrt{g}} + \alpha\sqrt{\frac{g}{\pi}}\exp\left(-\frac{\alpha^2}{4g}\right), \quad 0 < g$
805.	$\frac{1-\exp(-\alpha\sqrt{p})}{p(p+\gamma)}$	$\frac{e^{-\gamma g}}{2\gamma}\left[\exp(-\alpha\sqrt{-\gamma})\,\text{erfc}\left(\frac{\alpha}{2\sqrt{g}}-\sqrt{-\gamma g}\right)+\exp(\alpha\sqrt{-\gamma})\,\text{erfc}\left(\frac{\alpha}{2\sqrt{g}}+\sqrt{-\gamma g}\right)\right]+\frac{1}{\gamma}\text{erf}\,\frac{\alpha}{2\sqrt{g}}-\frac{1}{\gamma}e^{-\gamma g}, \quad 0 < g$
806.	$\sqrt{p}\exp(-\alpha\sqrt{p})$	$\left(\frac{\alpha^2}{2g}-1\right)\frac{1}{2g\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right), \quad 0 < g$
807.	$\frac{1}{\sqrt{p}}\exp(-\alpha\sqrt{p})$	$\frac{1}{\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}\right), \quad 0 < g$
808.*	$\frac{1}{\alpha p\sqrt{p}}\exp(-\alpha\sqrt{p})+\frac{1}{p}$	$\frac{2}{\alpha}\sqrt{\frac{g}{\pi}}\exp\left(-\frac{\alpha^2}{4g}\right)+\text{erf}\,\frac{\alpha}{2\sqrt{g}}, \quad 0 < g$
809.	$\frac{\exp(-\alpha\sqrt{p})}{1+\sqrt{\beta p}}$	$\frac{1}{\sqrt{\pi\beta g}}\exp\left(-\frac{\alpha^2}{4g}\right)-\frac{1}{\beta}\exp\frac{\alpha\sqrt{\beta}+g}{\beta} \times \text{erfc}\,\frac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}}, \quad 0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
810.	$\dfrac{p \exp(-\alpha\sqrt{p})}{1+\sqrt{\beta p}}$	$\dfrac{\alpha^2\beta - 2(\beta+\alpha\sqrt{\beta})g + 4g^2}{4\beta g^2\sqrt{\pi\beta g}} \exp\left(-\dfrac{\alpha^2}{4g}\right) - \dfrac{1}{\beta^2} \exp \dfrac{\alpha\sqrt{\beta}+g}{\beta} \operatorname{erfc} \dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0 < g$
811.	$\dfrac{\exp(-\alpha\sqrt{p})}{p(1+\sqrt{\beta p})} - \dfrac{1}{p}$	$-\operatorname{erf} \dfrac{\alpha}{2\sqrt{g}} - \exp \dfrac{\alpha\sqrt{\beta}+g}{\beta} \times \operatorname{erfc} \dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0 < g$
812.	$\dfrac{\exp(-\alpha\sqrt{p})}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{\exp(-\alpha\sqrt{-\gamma}-\gamma g)}{2(1+\sqrt{-\beta\gamma})} \operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}} - \sqrt{-\gamma g}\right) + \dfrac{\exp(\alpha\sqrt{-\gamma}-\gamma g)}{2(1-\sqrt{-\beta\gamma})} \times \operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}} + \sqrt{-\gamma g}\right) - \dfrac{1}{1+\beta\gamma} \exp \dfrac{\alpha\sqrt{\beta}+g}{\beta} \times \operatorname{erfc} \dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}},\quad 0 < g$
813.	$\dfrac{p \exp(-\alpha\sqrt{p})}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{-\gamma \exp(-\alpha\sqrt{-\gamma}-\gamma g)}{2(1+\sqrt{-\beta\gamma})} \times \operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}} - \sqrt{-\gamma g}\right) + \dfrac{-\gamma \exp(\alpha\sqrt{-\gamma}-\gamma g)}{2(1-\sqrt{-\beta\gamma})} \times \operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}} + \sqrt{-\gamma g}\right) - \dfrac{1}{\beta(1+\beta\gamma)} \exp \dfrac{\alpha\sqrt{\beta}+g}{\beta} \times \operatorname{erfc} \dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}} + \dfrac{1}{\sqrt{\pi\beta g}} \exp\left(-\dfrac{\alpha^2}{4g}\right),\quad 0 < g$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
814.	$\dfrac{\sqrt{p}\exp(-\alpha\sqrt{p})}{1+\sqrt{\beta p}}$	$\dfrac{\alpha\sqrt{\beta}-2g}{2\beta g\sqrt{\pi g}}\exp\left(-\dfrac{\alpha^2}{4g}\right) + \dfrac{1}{\beta\sqrt{\beta}}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta}\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}}$, $0 < g$
815.	$\dfrac{\exp(-\alpha\sqrt{p})}{\sqrt{p}(1+\sqrt{\beta p})}$	$\dfrac{1}{\sqrt{\beta}}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta}\operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}}$, $0 < g$
816.	$\dfrac{\sqrt{p}\exp(-\alpha\sqrt{p})}{(p+\gamma)(1+\sqrt{\beta p})}$	$\dfrac{\sqrt{-\gamma}\exp(-\alpha\sqrt{-\gamma}-\gamma g)}{2(1+\sqrt{-\beta\gamma})} \times \operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}-\sqrt{-\gamma g}\right) - \dfrac{\sqrt{-\gamma}\exp(\alpha\sqrt{-\gamma}-\gamma g)}{2(1-\sqrt{-\beta\gamma})} \times \operatorname{erfc}\left(\dfrac{\alpha}{2\sqrt{g}}+\sqrt{-\gamma g}\right) + \dfrac{1}{\sqrt{\beta}(1+\beta\gamma)}\exp\dfrac{\alpha\sqrt{\beta}+g}{\beta} \times \operatorname{erfc}\dfrac{\alpha\sqrt{\beta}+2g}{2\sqrt{\beta g}}$, $0 < g$
817.	$\exp(-\alpha\sqrt{p+\beta})$	$\dfrac{\alpha}{2g\sqrt{\pi g}}\exp\left(-\beta g-\dfrac{\alpha^2}{4g}\right)$, $0 < g$
818.	$\dfrac{1}{p}\exp(-\alpha\sqrt{p+\beta}+\alpha\sqrt{\beta})-\dfrac{1}{p}$	$-\dfrac{1}{2}\left[\operatorname{erfc}\left(\sqrt{\beta g}-\dfrac{\alpha}{2\sqrt{g}}\right) - \exp(2\alpha\sqrt{\beta})\operatorname{erfc}\left(\sqrt{\beta g}+\dfrac{\alpha}{2\sqrt{g}}\right)\right]$, $0 < g$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
819.	$\dfrac{\exp(-\alpha\sqrt{p+\beta})}{p+\gamma}$	$\frac{1}{2}\Big[\exp(-\alpha\sqrt{\beta-\gamma}-\gamma g)\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}}-\sqrt{(\beta-\gamma)g}\right)+\exp(\alpha\sqrt{\beta-\gamma}-\gamma g)\times\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}}+\sqrt{(\beta-\gamma)g}\right)\Big],\quad 0<g$
820.	$\sqrt{p+\beta}\exp(-\alpha\sqrt{p+\beta})$	$\left(\frac{\alpha^2}{2g}-1\right)\frac{1}{2g\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}-\beta g\right),\quad 0<g$
821.	$\frac{1}{p}\sqrt{\frac{p}{\beta}+1}\exp(-\alpha\sqrt{p+\beta}+\alpha\sqrt{\beta})-\frac{1}{p}$	$\frac{1}{\sqrt{\pi\beta g}}\exp\left(-\frac{\alpha^2}{4g}-\beta g+\alpha\sqrt{\beta}\right)-\frac{1}{2}\Big[\operatorname{erfc}\left(\sqrt{\beta g}-\frac{\alpha}{2\sqrt{g}}\right)+\exp(2\alpha\sqrt{\beta})\operatorname{erfc}\left(\sqrt{\beta g}+\frac{\alpha}{2\sqrt{g}}\right)\Big],\quad 0<g$
822.	$\dfrac{\sqrt{p+\beta}\exp(-\alpha\sqrt{p+\beta})}{p+\gamma}$	$\frac{\sqrt{\beta-\gamma}}{2}\Big[\exp(-\alpha\sqrt{\beta-\gamma}-\gamma g)\times\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}}-\sqrt{(\beta-\gamma)g}\right)-\exp(\alpha\sqrt{\beta-\gamma}-\gamma g)\times\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{g}}+\sqrt{(\beta-\gamma)g}\right)\Big]+\frac{1}{\sqrt{\pi g}}\exp\left(-\frac{\alpha^2}{4g}-\beta g\right),\quad 0<g$
823.	$\dfrac{\exp(-\alpha\sqrt{p+\beta})}{\sqrt{p+\beta}}$	$\frac{1}{\sqrt{\pi g}}\exp\left(-\beta g-\frac{\alpha^2}{4g}\right),\quad 0<g$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
824.	$\dfrac{\exp(-\alpha\sqrt{p+\beta}+\alpha\sqrt{\beta})}{p\sqrt{\dfrac{p}{\beta}+1}}-\dfrac{1}{p}$	$-\dfrac{1}{2}\Big[\operatorname{erfc}\Big(\sqrt{\beta g}-\dfrac{\alpha}{2\sqrt{g}}\Big) + \exp(2\alpha\sqrt{\beta})\operatorname{erfc}\Big(\sqrt{\beta g}+\dfrac{\alpha}{2\sqrt{g}}\Big)\Big],$ $0 < g$
825.	$\dfrac{\exp(-\alpha\sqrt{p+\beta})}{(p+\gamma)\sqrt{p+\beta}}$	$\dfrac{1}{2\sqrt{\beta-\gamma}}\Big[\exp(-\alpha\sqrt{\beta-\gamma}-\gamma g) \times\operatorname{erfc}\Big(\dfrac{\alpha}{2\sqrt{g}}-\sqrt{(\beta-\gamma)g}\Big) -\exp(\alpha\sqrt{\beta-\gamma}-\gamma g) \times\operatorname{erfc}\Big(\dfrac{\alpha}{2\sqrt{g}}+\sqrt{(\beta-\gamma)g}\Big)\Big],$ $0 < g$
841.	$\dfrac{1}{\sqrt{\lvert p\rvert}}\exp(-\rho\sqrt{\lvert p\rvert})$	$\sqrt{\dfrac{2}{\pi\lvert g\rvert}}\Big[\sin\dfrac{\rho^2}{4\lvert g\rvert}C\Big(\dfrac{\rho}{\sqrt{2\pi\lvert g\rvert}}\Big) -\cos\dfrac{\rho^2}{4\lvert g\rvert}S\Big(\dfrac{\rho}{\sqrt{2\pi\lvert g\rvert}}\Big)\Big] +\dfrac{1}{\sqrt{\pi\lvert g\rvert}}\cos\Big(\dfrac{\rho^2}{4\lvert g\rvert}+\dfrac{\pi}{4}\Big)$
842.	$\dfrac{1}{\sqrt{\lvert p\rvert}}\exp(-\rho\sqrt{\lvert p\rvert}-\sigma\lvert p\rvert)$	$\dfrac{1}{2\sqrt{\pi(\sigma+ig)}}\exp\dfrac{\rho^2}{4(\sigma+ig)}\operatorname{erfc}\dfrac{\rho}{2\sqrt{\sigma+ig}} +\dfrac{1}{2\sqrt{\pi(\sigma-ig)}}\exp\dfrac{\rho^2}{4(\sigma-ig)} \times\operatorname{erfc}\dfrac{\rho}{2\sqrt{\sigma-ig}}$
843.*	$\sqrt{\lvert p\rvert}\sin(a\sqrt{\lvert p\rvert})$	$-\dfrac{a}{2\pi g^2}-\dfrac{1}{\lvert g\rvert\sqrt{2\pi\lvert g\rvert}}\Big[\Big(\cos\dfrac{a^2}{4\lvert g\rvert} -\dfrac{a^2}{2\lvert g\rvert}\sin\dfrac{a^2}{4\lvert g\rvert}\Big)C\Big(\dfrac{a}{\sqrt{2\pi\lvert g\rvert}}\Big) +\Big(\sin\dfrac{a^2}{4\lvert g\rvert}+\dfrac{a^2}{2\lvert g\rvert}\cos\dfrac{a^2}{4\lvert g\rvert}\Big) \times S\Big(\dfrac{a}{\sqrt{2\pi\lvert g\rvert}}\Big)\Big]$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
844.	$\dfrac{\sqrt{\lvert p\rvert}\sin(a\sqrt{\lvert p\rvert})}{p}$	$\pm\sqrt{\dfrac{2}{\pi\lvert g\rvert}}\left[\sin\dfrac{a^2}{4\lvert g\rvert}S\left(\dfrac{a}{\sqrt{2\pi\lvert g\rvert}}\right) + \cos\dfrac{a^2}{4\lvert g\rvert}C\left(\dfrac{a}{\sqrt{2\pi\lvert g\rvert}}\right)\right], \quad 0 < \pm g$
845.	$\cos(\alpha\sqrt{\lvert p\rvert})\,e^{-\beta\lvert p\rvert}$	$\dfrac{\beta}{\pi(\beta^2+g^2)} + \dfrac{i\alpha}{4(\beta+ig)\sqrt{\pi(\beta+ig)}} \times \exp\left[-\dfrac{\alpha^2}{4(\beta+ig)}\right]\operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta+ig}} + \dfrac{i\alpha}{4(\beta-ig)\sqrt{\pi(\beta-ig)}} \times \exp\left[-\dfrac{\alpha^2}{4(\beta-ig)}\right]\operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta-ig}}$
846.	$\dfrac{\sin(\alpha\sqrt{\lvert p\rvert})\,e^{-\beta\lvert p\rvert}}{\sqrt{\lvert p\rvert}}$	$\dfrac{1}{2i\sqrt{\pi(\beta+ig)}}\exp\left[-\dfrac{\alpha^2}{4(\beta+ig)}\right] \times \operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta+ig}} + \dfrac{1}{2i\sqrt{\pi(\beta-ig)}} \times \exp\left[-\dfrac{\alpha^2}{4(\beta-ig)}\right]\operatorname{erf}\dfrac{i\alpha}{2\sqrt{\beta-ig}}$
861.	$\dfrac{\exp\left[-c\sqrt{p(p+\alpha)}\right]}{\sqrt{p(p+\alpha)}}$	$e^{-\frac{1}{2}\alpha g}I_0\left(\dfrac{\alpha}{2}\sqrt{g^2-c^2}\right), \quad c < g$
862.*	$\sqrt{\dfrac{p}{p+\alpha}}\exp\left[-c\sqrt{p(p+\alpha)}\right]$	$e^{-\frac{1}{2}\alpha c}\mathbf{S}_0(g-c) + \dfrac{\alpha}{2}e^{-\frac{1}{2}\alpha g}\left[\dfrac{g}{\sqrt{g^2-c^2}}I_1\left(\dfrac{\alpha}{2}\sqrt{g^2-c^2}\right) - I_0\left(\dfrac{\alpha}{2}\sqrt{g^2-c^2}\right)\right], \quad c < g$
863.*	$\exp\left[-c\sqrt{(p+\alpha)(p+\beta)}\right]$	$e^{-\frac{1}{2}(\alpha+\beta)c}\mathbf{S}_0(g-c) + \dfrac{c(\alpha-\beta)}{2\sqrt{g^2-c^2}}e^{-\frac{1}{2}(\alpha+\beta)g}I_1\left(\dfrac{\alpha-\beta}{2} \times \sqrt{g^2-c^2}\right), \quad c < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
864.*	$\sqrt{\frac{p+\alpha}{p+\beta}}\exp\left[-c\sqrt{(p+\alpha)(p+\beta)}\right]$	$e^{-\frac{1}{2}(\alpha+\beta)c}\mathbf{S}_0(g-c) + \frac{\alpha-\beta}{2}e^{-\frac{1}{2}(\alpha+\beta)g}\left[\frac{g}{\sqrt{g^2-c^2}}I_1\left(\frac{\alpha-\beta}{2}\times\sqrt{g^2-c^2}\right)+I_0\left(\frac{\alpha-\beta}{2}\sqrt{g^2-c^2}\right)\right]$, $c < g$
865.*	$\exp(-c\sqrt{p^2+a^2})$	$\mathbf{S}_0(g-c) - \frac{ac}{\sqrt{g^2-c^2}}J_1(a\sqrt{g^2-c^2})$, $c < g$
866.	$\frac{\exp(-c\sqrt{p^2+a^2})}{\sqrt{p^2+a^2}}$	$J_0(a\sqrt{g^2-c^2})$, $c < g$
867.	$\exp(-\alpha\sqrt{\beta^2-p^2})$	$\frac{\alpha\beta K_1(\beta\sqrt{\alpha^2+g^2})}{\pi\sqrt{\alpha^2+g^2}}$
868.	$\frac{\exp(-\alpha\sqrt{\beta^2-p^2})}{\sqrt{\beta^2-p^2}}$	$\frac{1}{\pi}K_0(\beta\sqrt{\alpha^2+g^2})$
869.	$\frac{(\sqrt{p^2+a^2}+p)^{-\nu}}{\sqrt{p^2+a^2}}\exp(-r\sqrt{p^2+a^2})$	$a^{-\nu}\left(\frac{g-r}{g+r}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{g^2-r^2})$, $r < g$
871.*	$\cos(a\sqrt{\beta^2-p^2})$	$\frac{1}{2}\mathbf{S}_0(g+a)+\frac{1}{2}\mathbf{S}_0(g-a) - \frac{a\beta J_1(\beta\sqrt{a^2-g^2})}{2\sqrt{a^2-g^2}}$, $-a < g < a$
872.	$\frac{\sin(a\sqrt{\beta^2-p^2})}{\sqrt{\beta^2-p^2}}$	$\frac{1}{2}J_0(\beta\sqrt{a^2-g^2})$, $-a < g < a$
881.	$\tan^{-1}\frac{r}{p+\rho}$	$\frac{1}{g}e^{-\rho g}\sin rg$, $0 < g$
891.	$(p+\beta)^{-\alpha}\log(p+\beta)$	$\frac{1}{\Gamma(\alpha)}g^{\alpha-1}e^{-\beta g}[\psi(\alpha)-\log g]$, $0 < g$

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
892.	$p^{-\alpha}\log p$,	$0 < R(\alpha) < 1$	$\frac{\psi(\alpha) - \log g}{\Gamma(\alpha)} g^{\alpha-1}$,	$0 < g$
893.*	$\frac{\log p}{p} = \lim_{\beta\to 0} \frac{\log(p+\beta)}{p+\beta}$		$\psi(1) - \log g$,	$0 < g$
894.	$\log \frac{p+\gamma}{p+\beta}$		$\frac{1}{g}(e^{-\beta g} - e^{-\gamma g})$,	$0 < g$

Part 9. Other Transcendental Functions of f.

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation		Coefficient $G(g)$ for the Unit Impulse	
901.	$\exp p^2 \operatorname{erfc} p$		$\frac{1}{\sqrt{\pi}} \exp(-\frac{1}{4}g^2)$,	$0 < g$
902.	$\frac{1}{\sqrt{p}} \exp\left(\frac{1}{p}\right) \operatorname{erfc}\left(\frac{1}{\sqrt{p}}\right)$		$\frac{1}{\sqrt{\pi g}} \exp(-2\sqrt{g})$,	$0 < g$
911.	$p^{\frac{1}{2}-\alpha} K_{\frac{1}{2}-\alpha}(ap)$,	$0 < R(\alpha) < 1$	$\frac{\sqrt{\pi}}{\Gamma(\alpha)} (2a)^{\frac{1}{2}-\alpha} (g^2 - a^2)^{\alpha-1}$,	$a < g$
912.	$K_0(ap)$		$\frac{1}{\sqrt{g^2 - a^2}}$,	$a < g$
913.*	$\frac{1}{p} K_0(ap) = \lim_{\beta\to 0} \frac{K_0[a(p+\beta)]}{p+\beta}$		$\cosh^{-1} \frac{g}{a}$,	$a < g$
914.	$p^{\frac{1}{2}-\alpha} I_{\alpha-\frac{1}{2}}(ap)$		$\frac{1}{\sqrt{\pi}\,\Gamma(\alpha)} (2a)^{\frac{1}{2}-\alpha} (a^2 - g^2)^{\alpha-1}$,	$-a < g < a$
915.	$(\beta^2 - p^2)^{\frac{1}{2}\nu} K_\nu(\sqrt{\beta^2 - p^2})$		$\frac{1}{\sqrt{2\pi}} \left(\frac{\beta^2}{g^2+1}\right)^{\frac{1}{2}\nu+\frac{1}{4}} K_{\nu+\frac{1}{2}}(\beta\sqrt{g^2+1})$	
916.	$(\rho^2 + f^2)^{-\frac{1}{4}} K_{\frac{1}{4}}(2\pi\rho\sqrt{\rho^2 + f^2})$		$(\rho^2 + g^2)^{-\frac{1}{4}} K_{\frac{1}{4}}(2\pi\rho\sqrt{\rho^2 + g^2})$	
917.	$K_0(\beta\sqrt{\rho^2 - p^2})$		$\frac{\exp(-\rho\sqrt{g^2 + \beta^2})}{2(g^2 + \beta^2)^{\frac{1}{2}}}$	

* A star marks a pair as being the limit approached by regular pairs.

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse	
918.	$K_0(\alpha\|p\|)$	$\dfrac{1}{2\sqrt{\alpha^2 + g^2}}$	
919.	$\|p\|K_1(\alpha\|p\|)$	$\dfrac{\alpha}{2\sqrt{(\alpha^2 + g^2)^3}}$	
920.	$K_0(\alpha\sqrt{p})$	$\dfrac{1}{2g}\exp\left(-\dfrac{\alpha^2}{4g}\right),$	$0 < g$
921.	$\sqrt{p}\,K_1(\alpha\sqrt{p})$	$\dfrac{\alpha}{4g^2}\exp\left(-\dfrac{\alpha^2}{4g}\right),$	$0 < g$
922.	$\dfrac{\alpha}{\sqrt{p}}K_1(\alpha\sqrt{p}) - \dfrac{1}{p}$	$\exp\left(-\dfrac{\alpha^2}{4g}\right) - 1,$	$0 < g$
923.	$\sqrt{p}\exp(p^2)K_{\frac14}(p^2)$	$\dfrac{1}{\sqrt{2g}}\exp(-\tfrac18 g^2),$	$0 < g$
924.	$\dfrac{1}{\sqrt{p}}\exp\left(\dfrac{1}{p}\right)K_{\frac14}\left(\dfrac{1}{p}\right)$	$(2g)^{-\frac34}\exp(-2\sqrt{2g}),$	$0 < g$
925.	$\dfrac{1}{\sqrt{p}}\exp\left(-\dfrac{1}{p}\right)I_{\alpha-\frac12}\left(\dfrac{1}{p}\right)$	$\dfrac{1}{\sqrt{\pi g}}J_{2\alpha-1}(2\sqrt{2g}),$	$0 < g$
931.	$\sqrt{p}K_{\nu+\frac14}(p)K_{\nu-\frac14}(p), \quad -\tfrac34 < R(\nu) < \tfrac34$	$\dfrac{\sqrt{\pi}(y^{2\nu} + y^{-2\nu})}{\sqrt{2g(g^2-4)}}, \quad y = \tfrac12(g + \sqrt{g^2-4}),$	$2 < g$
932.	$\sqrt{p}\left[I_{\nu-x}(p)I_{-\nu-x}(p)\right]_{x=-\frac14}^{x=\frac14}$	$\dfrac{\sqrt{2}(y^{2\nu} + y^{-2\nu})}{\pi\sqrt{\pi g(4-g^2)}}, \quad y = \tfrac12(g + \sqrt{g^2-4}),$	$0 < g < 2$
933.	$\sqrt{p}I_v(p)K_{v+\frac12}(p)$	$\dfrac{(-1)^v(x^{2v+\frac12} + x^{-2v-\frac12})}{\sqrt{2\pi g(4-g^2)}},$ $x = \tfrac12(g + \sqrt{g^2-4}),$	$0 < g < 2$

TABLE I (Continued)

Pair No.	Coefficient $F(f)$ for the Cisoidal Oscillation	Coefficient $G(g)$ for the Unit Impulse
934.	$\sqrt{p}\left[J_{-\frac{1}{2}+x+\alpha}(p)\,Y_{-\frac{1}{2}-x+\alpha}(p)\right]_{x=-\frac{1}{2}}^{x=\frac{1}{2}}$	$\dfrac{2^{2\alpha}(g+\sqrt{g^2+4})^{\frac{1}{2}-2\alpha}}{\pi\sqrt{\pi g(g^2+4)}}$, $0 < g$
935.	$J_{\alpha-\frac{1}{2}}(\sqrt{\delta^2-p^2}+ip)J_{\alpha-\frac{1}{2}}(\sqrt{\delta^2-p^2}-ip)$	$\frac{1}{\pi}(4-g^2)^{-\frac{1}{2}}J_{2\alpha-1}(\delta\sqrt{4-g^2})$, $-2 < g < 2$
936.	$I_{\alpha-\frac{1}{2}}(\sqrt{p^2+b^2}-p)K_{\alpha-\frac{1}{2}}(\sqrt{p^2+b^2}+p)$	$(g^2-4)^{-\frac{1}{2}}J_{2\alpha-1}(b\sqrt{g^2-4})$, $2 < g$
981.*	$\mathfrak{S}_0(f) = \frac{1}{\pi}\lim_{\beta\to 0}\frac{\beta}{\beta^2+f^2}$	1
982.*	$\mathfrak{S}_0(f-f_0)$, lim by 981*	$\text{cis}(2\pi f_0 g)$
983.*	$\mathfrak{S}_0(f-f_0)+\mathfrak{S}_0(f+f_0)$, lim by 981*	$2\cos(2\pi f_0 g)$
984.*	$\mathfrak{S}_0(f-f_0)-\mathfrak{S}_0(f+f_0)$, lim by 981*	$i2\sin(2\pi f_0 g)$
985.*	$\mathfrak{S}_1(f) = \lim_{\beta\to 0}\left(-\frac{2\pi f}{\beta\sqrt{\beta}}e^{-\pi f^2/\beta}\right)$	$-i2\pi g$
986.*	$\mathfrak{S}_{-1}(f) = \lim_{\beta\to 0}\left(\lambda \pm \frac{1}{2}\right)e^{-\beta\lvert f\rvert}$, $0 < \pm f$	$-\frac{1}{i2\pi g}+\lambda\mathfrak{S}_0(g)$
987.*	$\mathfrak{S}_{-2}(f) = \lim_{\beta\to 0}\left(\frac{1}{2}\lvert f\rvert+\lambda f+\mu\right)e^{-\beta\lvert f\rvert}$	$-\frac{1}{4\pi^2 g^2}+\frac{\lambda}{i2\pi}\mathfrak{S}_1(g)+\mu\mathfrak{S}_0(g)$

* A star marks a pair as being the limit approached by regular pairs.

TABLE II—ADMITTANCES OF

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse $= \mathfrak{S}_0(t)$ Effect: $\mathfrak{M}Y(p)$ 403*
1	$Y(p) = \dfrac{Cp + G}{LC(p - p_1)(p - p_2)}$ Inductance L and resistance R in series with parallel combination of capacity C and conductance G. Cause: Voltage across terminals. Effect: Current through network. $\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \dfrac{-(RC + LG) \pm \Delta}{2LC}$, $\Delta = \sqrt{(RC - LG)^2 - 4LC}$.	$\dfrac{1}{\Delta}[(Cp_1 + G)e^{p_1 t} - (Cp_2 + G)e^{p_2 t}]$, $0 < t$ 448, 449
2	$Y(p) = Cp + G$ Same as 1, except $R = 0$, $L = 0$.	$C\mathfrak{S}_1(t) + G\mathfrak{S}_0(t)$ 403*, 404*
3	$Y(p) = \exp\left[-\dfrac{x}{v}\sqrt{(p + \rho)^2 - \sigma^2}\right]$ Semi-infinite smooth line (resistance R, inductance L, conductance G, and capacity C per unit length). Cause: Initial voltage. Effect: Voltage at distance x from end. $v = (LC)^{-\frac{1}{2}}$, $k = (L/C)^{\frac{1}{2}}$, $\alpha = R/(2L)$, $\beta = G/(2C)$, $\rho = \alpha + \beta$, $\sigma = \alpha - \beta$, $z = \sqrt{t^2 - (x/v)^2}$.	$\exp\left(-\dfrac{\rho x}{v}\right)\mathfrak{S}_0\left(t - \dfrac{x}{v}\right) + \dfrac{\sigma x}{vz}e^{-\rho t} I_1(\sigma z)$, $\dfrac{x}{v} < t$ 863*
4	$Y(p) = \dfrac{1}{k}\sqrt{\dfrac{p + \rho - \sigma}{p + \rho + \sigma}} \times \exp\left[-\dfrac{x}{v}\sqrt{(p + \rho)^2 - \sigma^2}\right]$ Same as 3, except Effect: Current at distance x from end.	$\dfrac{1}{k}\exp\left(-\dfrac{\rho x}{v}\right)\mathfrak{S}_0\left(t - \dfrac{x}{v}\right) + \dfrac{1}{k}e^{-\rho t}\left[\dfrac{\sigma t}{z}I_1(\sigma z) - \sigma I_0(\sigma z)\right]$, $\dfrac{x}{v} < t$ 864*

AND TRANSIENTS IN PHYSICAL SYSTEMS

Cause: Unit Step (0, 1) $= \mathfrak{S}_{-1}(t)$, $\lambda = \frac{1}{2}$ Effect: $\mathfrak{M}[Y(p)/p]$ 415*	Cause: Unit Cisoid × Unit Step (0, 1) $= e^{p_0 t}\mathfrak{S}_{-1}(t)$, $\lambda = \frac{1}{2}$ Effect: $\mathfrak{M}[Y(p)/(p - p_0)]$ 440*
$\frac{1}{R + G^{-1}} + \frac{Cp_1 + G}{\Delta p_1} e^{p_1 t} - \frac{Cp_2 + G}{\Delta p_2} e^{p_2 t}, \quad 0 < t$ 448, 454, 415*	$\frac{(Cp_0 + G)e^{p_0 t}}{LC(p_0 - p_1)(p_0 - p_2)} + \frac{(Cp_1 + G)e^{p_1 t}}{\Delta(p_1 - p_0)} - \frac{(Cp_2 + G)e^{p_2 t}}{\Delta(p_2 - p_0)}, \quad 0 < t$ 452, 453
$C\mathfrak{S}_0(t) + G, \quad 0 < t$ 403*, 415*	$C\mathfrak{S}_0(t) + (Cp_0 + G)e^{p_0 t}, \quad 0 < t$ 438, 441*

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}\,Y(p)$
5	$Y(p) = \exp(-y\sqrt{p+2\beta})$ Same as 3, except $L = 0$. $y = x\sqrt{RC}$.	$\frac{y}{2t\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right), \quad 0 < t$ 817
6	$Y(p) = u\sqrt{p+2\beta}\exp(-y\sqrt{p+2\beta})$ Same as 4, except $L = 0$. $y = x\sqrt{RC}, \quad u = \sqrt{C/R}$.	$\frac{u(y^2-2t)}{4t^2\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right), 0 < t$ 820
7	$Y(p) = \frac{\exp(-y\sqrt{p+2\beta})}{u\sqrt{p+2\beta}}$ Same as 3, except $L = 0$ and Cause: Initial current. $y = x\sqrt{RC}, \quad u = \sqrt{C/R}$.	$\frac{1}{u\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right), \quad 0 < t$ 823
8	$Y(p) = \frac{1}{k}\sqrt{\frac{p}{p+2\alpha}}$ $\times \exp\left[-\frac{x}{v}\sqrt{p(p+2\alpha)}\right]$ Same as 4, except $G = 0$.	$\frac{1}{k}\exp\left(-\frac{\alpha x}{v}\right)\mathfrak{S}_0\left(t - \frac{x}{v}\right)$ $+\frac{1}{k}e^{-\alpha t}\left[\frac{\alpha t}{z}I_1(\alpha z) - \alpha I_0(\alpha z)\right], \frac{x}{v} < t$ 862*
9	$Y(p) = k\sqrt{\frac{p+2\alpha}{p}}$ Same as 3, except $G = 0$, $x = 0$, and Cause: Initial current.	$k\mathfrak{S}_0(t) + k\alpha e^{-\alpha t}[I_1(\alpha t) + I_0(\alpha t)]$, $0 < t$ 553*

Continued

Cause: Unit Step (0, 1) Effect: $\mathcal{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step (0, 1) Effect: $\mathcal{M}[Y(p)/(p - p_0)]$
$\frac{1}{2}\Big[\exp(-y\sqrt{2\beta})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{2\beta t}\right)$ $+ \exp(y\sqrt{2\beta})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{2\beta t}\right)\Big],$ $0 < t$ 818, 415*	$\frac{1}{2}e^{p_0 t}\Big[\exp(-y\sqrt{2\beta + p_0})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(2\beta + p_0)t}\right)$ $+ \exp(y\sqrt{2\beta + p_0})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(2\beta + p_0)t}\right)\Big], \quad 0 < t$ 819
$\frac{u}{\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right) + \frac{u\sqrt{2\beta}}{2}$ $\times\Big[\exp(-y\sqrt{2\beta})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{2\beta t}\right)$ $- \exp(y\sqrt{2\beta})\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{2\beta t}\right)\Big],$ $0 < t$ 821, 415*	$\frac{u}{\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - 2\beta t\right)$ $+ \frac{u\sqrt{2\beta + p_0}}{2}e^{p_0 t}\Big[\exp(-y\sqrt{2\beta + p_0})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(2\beta + p_0)t}\right)$ $- \exp(y\sqrt{2\beta + p_0})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(2\beta + p_0)t}\right)\Big], \quad 0 < t$ 822
$\frac{1}{2u\sqrt{2\beta}}\Big[\exp(-y\sqrt{2\beta})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{2\beta t}\right) - \exp(y\sqrt{2\beta})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{2\beta t}\right)\Big], \quad 0 < t$ 824, 415*	$\frac{e^{p_0 t}}{2u\sqrt{2\beta + p_0}}\Big[\exp(-y\sqrt{2\beta + p_0})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(2\beta + p_0)t}\right)$ $- \exp(y\sqrt{2\beta + p_0})$ $\times\,\text{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(2\beta + p_0)t}\right)\Big], \quad 0 < t$ 825
$\frac{1}{k}e^{-\alpha t}I_0(\alpha z), \quad \frac{x}{v} < t$ 861	
$ke^{-\alpha t}[2\alpha t I_1(\alpha t)$ $+ (1 + 2\alpha t)I_0(\alpha t)], \quad 0 < t$ 554*	

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}Y(p)$
10	$Y(p) = \exp\left(-\frac{\rho x}{v} - \frac{x}{v}p\right)$ Same as 3, except $R/L = G/C$.	$\exp\left(-\frac{\rho x}{v}\right)\mathfrak{S}_0\left(t - \frac{x}{v}\right)$ 601*
11	$Y(p) = \frac{\sqrt{p + 2\alpha}}{\sqrt{p} + \sqrt{p + 2\alpha}}$ Semi-infinite smooth line (resistance R, inductance L, and capacity C per unit length). Cause: Voltage applied through resistance $R_0 = \sqrt{L/C}$. Effect: Voltage at end of line. $\alpha = R/(2L)$.	$\frac{1}{2}\mathfrak{S}_0(t) + \frac{1}{2t}e^{-\alpha t}I_1(\alpha t), \quad 0 < t$ 559*
12	$Y(p) = \frac{\exp(-y\sqrt{p})}{1 + \sqrt{p/\lambda}}$ Semi-infinite smooth line (resistance R and capacity C per unit length). Cause: Voltage applied through resistance R_0. Effect: Voltage at distance x from end. $y = x\sqrt{RC}, \quad \lambda = R/(CR_0^2)$.	$\sqrt{\frac{\lambda}{\pi t}}\exp\left(-\frac{y^2}{4t}\right) - \lambda\exp(y\sqrt{\lambda} + \lambda t)$ $\times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 809
13	$Y(p) = \frac{u\sqrt{p}\exp(-y\sqrt{p})}{1 + \sqrt{p/\lambda}}$ Same as 12, except Effect: Current at distance x from end. $u = \sqrt{C/R}$.	$\frac{u(y - 2t\sqrt{\lambda})}{2t}\sqrt{\frac{\lambda}{\pi t}}\exp\left(-\frac{y^2}{4t}\right)$ $+ u\lambda\sqrt{\lambda}\exp(y\sqrt{\lambda} + \lambda t)$ $\times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{\lambda t}\right), \quad 0 < t$ 814
14	$Y(p) = \frac{u\sqrt{p}}{1 + \sqrt{p/\lambda}}$ Same as 13, except $x = 0$. $u = \sqrt{C/R}$.	$u\sqrt{\lambda}\left[\mathfrak{S}_0(t) - \sqrt{\frac{\lambda}{\pi t}} + \lambda e^{\lambda t}\operatorname{erfc}\sqrt{\lambda t}\right],$ $0 < t$ 550*

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\exp\left(-\frac{\rho x}{v}\right), \quad \frac{x}{v} < t$ 602*	$\exp\left[-\frac{x}{v}(\rho+p_0)+p_0t\right], \quad \frac{x}{v} < t$ 604
$1-\frac{1}{2}e^{-\alpha t}[I_0(\alpha t)+I_1(\alpha t)], \quad 0 < t$ 560, 415*	
$\operatorname{erfc}\frac{y}{2\sqrt{t}} - \exp(y\sqrt{\lambda}+\lambda t) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\lambda t}\right), \quad 0 < t$ 811, 415*	$\frac{1}{2}e^{p_0t}\left[\frac{\exp(-y\sqrt{p_0})}{1+\sqrt{p_0/\lambda}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{p_0t}\right) + \frac{\exp(y\sqrt{p_0})}{1-\sqrt{p_0/\lambda}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{p_0t}\right)\right] - \frac{1}{1-p_0/\lambda}\exp(y\sqrt{\lambda}+\lambda t) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\lambda t}\right), \quad 0 < t$ 812
$u\sqrt{\lambda}\exp(y\sqrt{\lambda}+\lambda t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\lambda t}\right), \quad 0 < t$ 815	$\frac{u\sqrt{p_0}}{2}e^{p_0t}\left[\frac{\exp(-y\sqrt{p_0})}{1+\sqrt{p_0/\lambda}} \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{p_0t}\right) - \frac{\exp(y\sqrt{p_0})}{1-\sqrt{p_0/\lambda}} \times \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{p_0t}\right)\right] + \frac{u\sqrt{\lambda}}{1-p_0/\lambda} \times \exp(y\sqrt{\lambda}+\lambda t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\lambda t}\right), \quad 0 < t$ 816
$u\sqrt{\lambda}e^{\lambda t}\operatorname{erfc}\sqrt{\lambda t}, \quad 0 < t$ 551	$\frac{u\sqrt{\lambda}}{1-p_0/\lambda}\left[\sqrt{\frac{p_0}{\lambda}}e^{p_0t}\operatorname{erf}\sqrt{p_0t} - \frac{p_0}{\lambda}e^{p_0t} + e^{\lambda t}\operatorname{erfc}\sqrt{\lambda t}\right], \quad 0 < t$ 552

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{N}Y(p)$
15	$Y(p) = \dfrac{C_0 p \exp(-y\sqrt{p})}{1+\sqrt{p/\mu}}$ Semi-infinite smooth line (resistance R and capacity C per unit length). Cause: Voltage applied through capacity C_0. Effect: Current at distance x from end. $y = x\sqrt{RC}, \quad \mu = C/(RC_0^2)$.	$\dfrac{C_0(y^2 - 2yt\sqrt{\mu} - 2t + 4\mu t^2)}{4t^2}\sqrt{\dfrac{\mu}{\pi t}} \times \exp\left(-\dfrac{y^2}{4t}\right) - C_0\mu^2 \exp(y\sqrt{\mu}+\mu t) \times \operatorname{erfc}\left(\dfrac{y}{2\sqrt{t}} + \sqrt{\mu t}\right), \quad 0 < t$ 810
16	$Y(p) = \dfrac{C_0 p}{1+\sqrt{p/\mu}}$ Same as 15, except $x = 0$.	$-C_0\mu \mathfrak{S}_0(t) + \dfrac{C_0(2\mu t - 1)}{2t}\sqrt{\dfrac{\mu}{\pi t}} - C_0\mu^2 e^{\mu t} \operatorname{erfc}\sqrt{\mu t}, \quad 0 < t$ 544*
17	$Y(p) = \dfrac{w^{2n+1}[\sqrt{(p+\lambda)^2+w^2}+(p+\lambda)]^{-2n}}{k\sqrt{(p+\lambda)^2+w^2}}$ Semi-infinite artificial line (series element: resistance R and inductance L; shunt element: conductance G and capacity C; $R/L = G/C$; mid-series termination). Cause: Applied voltage. Effect: Current in nth section. $k = (L/C)^{\frac{1}{2}}, \lambda = R/L = G/C, w = 2(LC)^{-\frac{1}{2}}$.	$\dfrac{2}{L} e^{-\lambda t} J_{2n}(wt), \quad 0 < t$ 575
18	$Y(p) = \dfrac{2(2\alpha)^n}{R}\sqrt{\dfrac{p}{p+2\alpha}} \times (\sqrt{p+2\alpha}+\sqrt{p})^{-2n}$ Semi-infinite artificial line (series element: resistance R; shunt element: capacity C; mid-series termination). Cause: Applied voltage. Effect: Current in nth section. $\alpha = 2/(RC)$.	$\dfrac{\alpha}{R} e^{-\alpha t}[I_{n-1}(\alpha t) - 2I_n(\alpha t) + I_{n+1}(\alpha t)], \quad 0 < t$ 573

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid $\times$ Unit Step (0, 1) Effect: $\mathfrak{M}[Y(p)/(p - p_0)]$
$C_0\sqrt{\frac{\mu}{\pi t}}\exp\left(-\frac{y^2}{4t}\right)$ $- C_0\mu\exp(y\sqrt{\mu}+\mu t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\mu t}\right),$ $0 < t$ 809	$\frac{1}{2}C_0p_0\, e^{p_0t}\left[\frac{\exp(-y\sqrt{p_0})}{1+\sqrt{p_0/\mu}}\right.$ $\times\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}-\sqrt{p_0t}\right)$ $\left.+\frac{\exp(y\sqrt{p_0})}{1-\sqrt{p_0/\mu}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{p_0t}\right)\right]$ $+ C_0\sqrt{\frac{\mu}{\pi t}}\exp\left(-\frac{y^2}{4t}\right) - \frac{C_0\mu}{1-p_0/\mu}$ $\times\exp(y\sqrt{\mu}+\mu t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}+\sqrt{\mu t}\right),$ 813 $\quad 0 < t$
$C_0\sqrt{\frac{\mu}{\pi t}} - C_0\mu e^{\mu t}\operatorname{erfc}\sqrt{\mu t}, \quad 0 < t$ 543	$C_0\sqrt{\frac{\mu}{\pi t}} + \frac{C_0\mu}{1-p_0/\mu}\left[\frac{p_0}{\mu}e^{p_0t} - \frac{p_0}{\mu}\sqrt{\frac{p_0}{\mu}}\,e^{p_0t}\right.$ $\left.\times\operatorname{erf}\sqrt{p_0t} - e^{\mu t}\operatorname{erfc}\sqrt{\mu t}\right], \quad 0 < t$ 545
$\frac{2}{R}e^{-\alpha t}I_n(\alpha t), \quad 0 < t$ 574	

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathcal{M}\, Y(p)$	
19	$Y(p) = \exp\left(x - \lvert x\rvert \sqrt{p^2+1}\right)$ Vertical atmospheric waves, axis of x vertically upwards, velocity $= 1$, height of homogeneous atmosphere $= \frac{1}{2}$. Cause: Vertical displacement at $x = 0$. Effect: Vertical displacement at time t of particle whose undisturbed position is x.	$e^x \mathfrak{S}_0(t - \lvert x\rvert) - \frac{\lvert x\rvert e^x}{\sqrt{t^2 - x^2}} \times J_1(\sqrt{t^2 - x^2})$, 865*	$\lvert x\rvert < t$
20	$Y(p) = \frac{\exp\left(x - \lvert x\rvert \sqrt{p^2+1}\right)}{2\sqrt{p^2+1}}$ Same as 19, except Cause: Vertical force at $x = 0$.	$\frac{e^x}{2} J_0(\sqrt{t^2 - x^2})$, 866	$\lvert x\rvert < t$
21	$Y(p) = \frac{\lvert y\rvert}{\pi r} \sqrt{p}\, K_1(r\sqrt{p})$ Flow of heat in infinite plane. Cause: Temperature impulse at origin, temperature maintained zero along x-axis, except at origin. Effect: Temperature of point with coördinates (x, y) at time t. $r = \sqrt{x^2 + y^2}$.	$\frac{\lvert y\rvert}{4\pi t^2} \exp\left(-\frac{r^2}{4t}\right)$, 921	$0 < t$
22	$Y(p) = \frac{1}{p}\left[1 - \exp\left(-y\sqrt{\frac{p}{\nu}}\right)\right]$ Horizontal oscillations of deep viscous fluid, axis of y vertical, bottom plane $y = 0$, kinematic coefficient of viscosity $= \nu$. Cause: Applied horizontal force. Effect: Displacement of particle at y at time t, y assumed small.	$\operatorname{erf} \frac{y}{2\sqrt{\nu t}}$, 803	$0 < t$
23	$Y(p) = \frac{1}{2\pi} K_0\left(\frac{rp}{c}\right)$ Water waves radiating from center in an unlimited sheet of uniform depth h, gravity constant $= g$. Cause: Pressure at the origin. Effect: Velocity potential at distance r, time t. $c^2 = gh$.	$\frac{1}{2\pi} \cdot \frac{1}{\sqrt{t^2 - \frac{r^2}{c^2}}}$, 912	$\frac{r}{c} < t$

Continued

Cause: Unit Step (0, 1) Effect: $\mathfrak{A}\mathfrak{l}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step (0, 1) Effect: $\mathfrak{A}\mathfrak{l}[Y(p)/(p - p_0)]$
$\frac{\lvert y\rvert}{\pi r^2} \exp\left(-\frac{r^2}{4t}\right)$, $0 < t$ 922, 415*	
$y\sqrt{\frac{t}{\nu\pi}} \exp\left(-\frac{y^2}{4\nu t}\right) - \frac{y^2}{2\nu} + \left(\frac{y^2}{2\nu} + t\right) \operatorname{erf} \frac{y}{2\sqrt{\nu t}}$, $0 < t$ 804*, 415*	$\frac{1}{p_0} e^{p_0 t} - \frac{1}{p_0} \operatorname{erf} \frac{y}{2\sqrt{\nu t}} - \frac{e^{p_0 t}}{2p_0}\left[\exp\left(-y\sqrt{\frac{p_0}{\nu}}\right) \times \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}} - \sqrt{p_0 t}\right) + \exp\left(y\sqrt{\frac{p_0}{\nu}}\right) \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}} + \sqrt{p_0 t}\right)\right]$, 805 $0 < t$
$\frac{1}{2\pi} \cosh^{-1} \frac{ct}{r}$, $\frac{r}{c} < t$ 913*	

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M} Y(p)$
24	$Y(p) = \dfrac{\sin[(\pi - y)p]}{\sin \pi p}$ Flow of electricity in thin plane infinite strip, axis of x along lower edge of strip, axis of y across, width of strip $= \pi$, upper edge ($y = \pi$) maintained at zero potential. Cause: Potential along x axis. Effect: Potential at point (x, y).	$\dfrac{1}{2\pi} \cdot \dfrac{\sin y}{\cosh x - \cos y}$ 615
25	$Y(p) = \dfrac{\cos[(\pi - y)p]}{\cos \pi p}$ Same as 24, except upper edge ($y = \pi$) is insulated.	$\dfrac{1}{\pi} \cdot \dfrac{\sin \frac{1}{2}y \cosh \frac{1}{2}x}{\cosh x - \cos y}$ 616
26	$Y(p) = \exp(\kappa t p^2)$ Linear flow of heat in infinite solid, diffusivity κ, axis of x in direction of flow. Cause: Initial temperature. Effect: Temperature at time t at point x.	$\dfrac{1}{2\sqrt{\pi \kappa t}} \exp\left(-\dfrac{x^2}{4\kappa t}\right)$ 701
27	$Y(p) = \cos(tp^2)$ Transverse oscillations of infinite elastic plate; x and y axes in the plate, but all points with same y coördinate have same displacement. Cause: Initial displacement. Effect: Displacement perpendicular to plate at time t of point whose coördinate is x.	$\dfrac{1}{2\sqrt{\pi t}} \sin\left(\dfrac{x^2}{4t} + \dfrac{\pi}{4}\right)$ 752

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\frac{1}{\pi}\tan^{-1}\frac{\tanh\frac{1}{2}x}{\tan\frac{1}{2}y}$ 617, 415*	
$\frac{1}{\pi}\tan^{-1}\frac{\sinh\frac{1}{2}x}{\sin\frac{1}{2}y}$ 618, 415*	
$\frac{1}{2}\operatorname{erf}\frac{x}{2\sqrt{\kappa t}}$ 727, 415*	$\frac{1}{2}\exp(\kappa t p_0^2 + p_0 x)\operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}} + p_0\sqrt{\kappa t}\right)$ 728, 440*
$\frac{1}{2}\left[S\left(\frac{x}{\sqrt{2\pi t}}\right) + C\left(\frac{x}{\sqrt{2\pi t}}\right)\right]$ 754, 415*	$\frac{1}{4}\exp(p_0 x + itp_0^2)\operatorname{erf}\left(\frac{x}{2\sqrt{it}} + p_0\sqrt{it}\right)$ $+\frac{1}{4}\exp(p_0 x - itp_0^2)$ $\times\operatorname{erf}\left(\frac{x}{2\sqrt{-it}} + p_0\sqrt{-it}\right)$ 755, 440*

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{A} Y(p)$
28	$Y(p) = \frac{\sin(tp^2)}{p^2}$ Same as 27, except Cause: Initial velocity.	$\sqrt{\frac{t}{\pi}} \sin\left(\frac{x^2}{4t} + \frac{\pi}{4}\right) + \frac{x}{2}\left[S\left(\frac{x}{\sqrt{2\pi t}}\right) - C\left(\frac{x}{\sqrt{2\pi t}}\right)\right]$ 756
29	$Y(p) = \cos(t\sqrt{1 - p^2})$ Same as 19, except Cause: Initial displacement multiplied by e^{-x}. Effect: Vertical displacement multiplied by e^{-x} at time t of particle whose undisturbed position is x.	$\frac{1}{2}[\mathfrak{S}_0(x - t) + \mathfrak{S}_0(x + t)] - \frac{tJ_1(\sqrt{t^2 - x^2})}{2\sqrt{t^2 - x^2}}, \quad -t < x < t$ 871*
30	$Y(p) = \frac{\sin(t\sqrt{1 - p^2})}{\sqrt{1 - p^2}}$ Same as 29, except Cause: Initial velocity multiplied by e^{-x}.	$\frac{1}{2}J_0(\sqrt{t^2 - x^2}), \quad -t < x < t$ 872
31	$Y(p) = e^{-\lvert yp \rvert}$ Flow of electricity in infinite thin plane, x and y axes in the plane. Cause: Potential along x axis. Effect: Potential at point (x, y).	$\frac{1}{\pi} \cdot \frac{\lvert y \rvert}{x^2 + y^2}$ 632
32	$Y(p) = \cosh(atp)$ Transverse motion of infinite stretched elastic string, axis of x along equilibrium position of string, velocity of propagation along string $= a$. Cause: Initial displacement. Effect: Normal displacement of particle at x at time t.	$\frac{1}{2}[\mathfrak{S}_0(x - at) + \mathfrak{S}_0(x + at)]$ 619*

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid $\times$ Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\frac{1}{2}\left[\left(t+\frac{x^2}{2}\right)S\left(\frac{x}{\sqrt{2\pi t}}\right)+\left(t-\frac{x^2}{2}\right) \times C\left(\frac{x}{\sqrt{2\pi t}}\right)+x\sqrt{\frac{t}{\pi}}\sin\left(\frac{x^2}{4t}+\frac{\pi}{4}\right)\right]$ 757, 415*	
$\frac{1}{\pi}\tan^{-1}\frac{x}{\lvert y\rvert}$ 633, 415*	
$\pm\frac{1}{2}, \quad at < \pm x$ 620, 415*	$\begin{cases} \pm\frac{1}{2}e^{p_0 x}\cosh(atp_0), & at < \pm x \\ \frac{1}{2}e^{p_0 x}\sinh(atp_0), & -at < x < at \end{cases}$ 621, 440*

TABLE II

No.	Admittance $Y(p)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}Y(p)$
33	$Y(p) = \frac{\sinh(atp)}{ap}$ Same as 32, except Cause: Initial velocity.	$\frac{1}{2a}, \quad -at < x < at$ 622
34	$Y(p) = \frac{1}{\rho}\cos(\alpha\sqrt{\lvert p\rvert})e^{y\lvert p\rvert}$ Waves on deep water, axis of y vertically upwards, axis of x in the surface, density $= \rho$, gravity constant $= g$, $y \leqq 0$. Cause: Initial surface-impulse along x axis. Effect: Velocity potential at time t at point (x, y). $h = \frac{\alpha}{\sqrt{2\pi\lvert x\rvert}}, \quad \alpha = t\sqrt{g}.$	$\frac{-y}{\pi\rho(x^2+y^2)} + \frac{i\alpha}{4\rho\sqrt{\pi}(-y+ix)^{3/2}} \times \exp\left[-\frac{\alpha^2}{4(-y+ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y+ix}} + \frac{i\alpha}{4\rho\sqrt{\pi}(-y-ix)^{3/2}} \times \exp\left[-\frac{\alpha^2}{4(-y-ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y-ix}}$ 845
35	$Y(p) = -\frac{\sqrt{\lvert p\rvert}}{\rho\sqrt{g}}\sin(\alpha\sqrt{\lvert p\rvert})$ Same as 34, except Effect: Surface elevation at time t at point x.	$\frac{\alpha}{2\pi\rho x^2\sqrt{g}} + \frac{1}{\rho\lvert x\rvert\sqrt{2\pi g\lvert x\rvert}} \times \{[\cos(\frac{1}{2}\pi h^2) - \pi h^2\sin(\frac{1}{2}\pi h^2)]C(h) + [\sin(\frac{1}{2}\pi h^2) + \pi h^2\cos(\frac{1}{2}\pi h^2)]S(h)\}$ 843*
36	$Y(p) = \sqrt{g}\frac{\sin(\alpha\sqrt{\lvert p\rvert})}{\sqrt{\lvert p\rvert}}e^{y\lvert p\rvert}$ Same as 34, except Cause: Initial surface elevation.	$\frac{\sqrt{g}}{2i\sqrt{\pi(-y+ix)}} \times \exp\left[-\frac{\alpha^2}{4(-y+ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y+ix}} + \frac{\sqrt{g}}{2i\sqrt{\pi(-y-ix)}} \times \exp\left[-\frac{\alpha^2}{4(-y-ix)}\right] \times \operatorname{erf}\frac{i\alpha}{2\sqrt{-y-ix}}$ 846

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/p]$	Cause: Unit Cisoid × Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}[Y(p)/(p-p_0)]$
$\begin{cases} \pm\frac{1}{2}t, & at < \pm x \\ \dfrac{x}{2a}, & -at < x < at \end{cases}$ 623, 415*	$\begin{cases} \pm\dfrac{1}{2ap_0}e^{p_0x}\sinh(atp_0), & at < \pm x \\ \dfrac{1}{2ap_0}[e^{p_0x}\cosh(atp_0) - 1], & -at < x < at \end{cases}$ 624, 440*
$\mp\dfrac{1}{\rho}\sqrt{\dfrac{2}{\pi g\|x\|}}[\sin(\frac{1}{2}\pi h^2)S(h) + \cos(\frac{1}{2}\pi h^2)C(h)], \quad 0 < \pm x$ 844	

TABLE II

No.	Admittance $Y(p_1, p_2)$ Illustrative System Cause and Effect	Cause: Unit Impulse Effect: $\mathfrak{M}_1\mathfrak{M}_2 Y(p_1, p_2)$
37	$Y(p_1, p_2) = \cos[t(p_1^2 + p_2^2)]$ Transverse oscillations of infinite elastic plate, x and y axes in the plate. Cause: Initial displacement. Effect: Displacement perpendicular to plate at time t of point whose coördinates are x and y.	$\frac{1}{4\pi t} \sin \frac{x^2 + y^2}{4t}$ 759, 758
38	$Y(p_1, p_2) = \exp(-z\sqrt{\lvert p_1^2 + p_2^2\rvert})$ Velocity potential function in semi-infinite incompressible fluid, x and y axes in surface of fluid, z extending down, $z \geqq 0$. Cause: Velocity potential at surface, $z = 0$. Effect: Velocity potential at point (x, y, z).	$\frac{1}{2\pi} \cdot \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$ 867, 919
39	$Y(p_1, p_2) = \frac{\exp(-z\sqrt{\lvert p_1^2 + p_2^2\rvert})}{\sqrt{\lvert p_1^2 + p_2^2\rvert}}$ Newtonian potential function in semi-infinite solid, x and y axes in face of solid, z extending into solid, $z \geqq 0$. Cause: Normal potential derivative at surface, $z = 0$. Effect: Potential at point (x, y, z).	$\frac{1}{2\pi} \cdot \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ 868, 918

Continued

Cause: Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}_1\mathfrak{M}_2[Y(p_1, p_2)/(p_1p_2)]$	Cause: Unit Cisoid × Unit Step $(-\frac{1}{2}, +\frac{1}{2})$ Effect: $\mathfrak{M}_1\mathfrak{M}_2 \dfrac{Y(p_1, p_2)}{(p_1-p_0)(p_2-p_0)}$
$\frac{1}{2}\left[S\left(\frac{x}{\sqrt{2\pi t}}\right)C\left(\frac{y}{\sqrt{2\pi t}}\right) + C\left(\frac{x}{\sqrt{2\pi t}}\right)S\left(\frac{y}{\sqrt{2\pi t}}\right)\right]$ 753; 754, 415*	
$\frac{1}{2\pi}\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}$ †	
$\frac{x}{4\pi}\log\frac{\sqrt{x^2+y^2+z^2}+y}{\sqrt{x^2+y^2+z^2}-y} + \frac{y}{4\pi}\log\frac{\sqrt{x^2+y^2+z^2}+x}{\sqrt{x^2+y^2+z^2}-x} - \frac{z}{2\pi}\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}$ †	

† This solution was obtained by double integration of the unit impulse solution, not by the operation indicated at the head of the column. The two pairs required for this operation nave not yet been found in closed form.